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ABSTRACT

This monograph chronicles a field-based investigation of children's learning in a second grade mathematics classroom over the course of a school year. The interrelated aspects of the processes by which children learn mathematics with meaning are illustrated here. The investigation was expanded to include a sociological analysis in an attempt to understand the relation between individual mathematical constructions and classroom social interactions. Data include clinical interviews of individual children's mathematical constructions and video recordings of classroom lessons. Microanalyses of specific episodes from the classroom are included in order to illustrate the theoretical constructs that evolved over the course of the investigation. The material is organized in three parts that relate to multiple perspectives of learning mathematics; culture, community, and the meaning of mathematics; and commentary on issues and reform in mathematics education. Contains 164 references. (DDR)

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Rethinking Elementary School Mathematics: Insights and Issues

edited by
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RETHINKING ELEMENTARY SCHOOL MATHEMATICS: INSIGHTS AND ISSUES

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RETHINKING ELEMENTARY SCHOOL MATHEMATICS:

INSIGHTS AND ISSUES

edited by

Terry Wood
Paul Cobb
Erna Yackel
Deborah Dillon

commentary by

Martin Simon

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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TABLE OF CONTENTS

ACKNOWLEDGMENTS iv

ABSTRACT v

AUTHORS vi

INTRODUCTION: BACKGROUND OF THE RESEARCH 1
Paul Cobb, Terry Wood, Erna Yackel, and Grayson Wheatley

PART 1: Learning Mathematics: Multiple Perspectives

1. Second-Grade Classroom: Psychological Perspective 7
Terry Wood (An overview of work by Cobb, Wood, Yackel, and Wheatley)

2. Creating an Environment For Learning Mathematics: Social Interaction Perspective 15
Terry Wood (An overview of work by Cobb, Yackel, and Wood)

3. Theoretical Orientation 21
Paul Cobb, Erna Yackel, and Terry Wood

4. Developing a Basis for Mathematical Communication Within Small Groups 33
Erna Yackel, Paul Cobb, and Terry Wood

5. The Relationship of Individual Children's Mathematical Conceptual Development to Small Group Interactions 45
Erna Yackel, Paul Cobb, and Terry Wood

6. The Nature of Whole-Class Discussion 55
Terry Wood, Paul Cobb, and Erna Yackel

PART 2: Meaning of Mathematics: Community and Culture

7. The Wider Social Context of Innovation in Mathematics Education 71
Deborah R. Dillon

PART 3: Commentary

8. Focus on Children's Mathematical Learning in Classrooms: Impact and Issues 99
Martin Simon

9. Context for Change: Themes Related to Mathematics Education Reform 109
Martin Simon

REFERENCES 115

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> If you are ever walking along a country road and see a turtle sitting on top of a fencepost, you have to believe that he had to have some help in getting there.
>
> *Author Unknown*

We are reminded of this saying as we reflect on the experiences we had as we conducted our research in the classroom. We are deeply indebted to the 20 children and their teacher who graciously let us become a part of their daily lives as we poked, prodded, and questioned them in an attempt to understand what it was like to be a member of their mathematics class.

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ABSTRACT

The intention of this monograph is to chronicle a field-based investigation of children's learning in one second-grade classroom over the course of the school year. In particular, the monograph illustrates the interrelated aspects of the processes by which children learn mathematics with meaning. Initially, the investigators intended to confine their investigation to cognitive analyses of children's learning in the classroom. However, it became apparent that a psychological perspective alone could not detail students' learning in the social setting of the classroom. Thus, the investigation was expanded to include a sociological analysis in an attempt to understand the relation between individual mathematical constructions and classroom social interactions.

The empirically grounded theoretical perspective was informed by qualitative methodology that extended from clinical interviews of individual children's mathematical constructions to video recordings of classroom lessons. Thus, the nature of children's mathematical learning was analyzed in clinical interviews and in pair-collaborative as well as whole-class settings. Microanalyses of specific episodes from the classroom are included in the monograph to illustrate the nature of the social interaction that occurred in the classroom. Moreover, these episodes are included to illustrate the theoretical constructs that evolved over the course of the investigation and that were used to explain the processes by which mathematical learning occurs within a classroom setting, along with the concomitant issues that arise for students and teachers.

Unintended ramifications of the project extended beyond the classroom to include the institution of the school as a part of the wider community. An interpretive stance also was used to inform an ethnographic investigation of the implications of change in the practices of elementary school mathematics on the wider sociopolitical setting of the community.

Finally, the results of the studies presented are considered in light of the current concerns and goals for reform in mathematics education in a concluding commentary.

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INTRODUCTION: BACKGROUND OF THE RESEARCH

Paul Cobb, Terry Wood, Erna Yackel, and Grayson Wheatley

Our primary goal in this monograph is to describe our investigation of children's learning of mathematics as it occurs within the social setting of the classroom. We have been working collaboratively for a number of years in second-grade mathematics classrooms. In this monograph we focus not only on the results of our work for one year in a second-grade classroom, but also on the evolution of our assumptions and theoretical orientations as we have tried to make sense of what it means to learn mathematics within the complexity of the classroom. As we began our collaboration, our intention was to investigate children's mathematical learning as they solved problem-centered activities and talked about their thinking. We initially took a psychological constructivist position (Piaget, 1970, 1980; von Glasersfeld, 1984, 1987) and focused exclusively on students' mathematical learning from a cognitive perspective. However, very early on in our work in the classroom, it became apparent that a psychological perspective alone could not account for children's learning as it occurred in the social setting of the classroom. We thus widened our purview to include a sociological perspective. In order to understand the relations between individual mathematical constructions and classroom social interactions, we complemented our cognitive orientation with constructs derived from symbolic interactionism (Bauersfeld, 1980; Blumer, 1969; Mead, 1934; Voigt, 1985) and ethnomethodology (Krummheuer, 1983; Mehan & Wood, 1975; Schutz, 1962).

As we began the project, we believed that the flexibility of the educational system beyond the classroom level was such that we could pursue our educational goals without finding ourselves in situations of conflict within the more encompassing situations of schooling. While we were aware that our evolving perspective about learning mathematics was different from the traditional position from which mathematics was taught, we did not fully realize the extent to which our ideas would influence the practice of elementary school mathematics. In this regard the classroom teacher was an invaluable colleague.

Although our intention was not to reform schools, we soon found our work compatible with issues and ideas proposed in the various documents calling for transformation of schools. We quickly learned that reform in mathematics education involved more than creating new problem-centered activities, introducing different instructional strategies, and helping teachers revise their pedagogical practices. As our project became more public, it soon became apparent that our work had ramifications that extended beyond the classroom to include the school as an institution embedded in the wider community (Erickson, 1986).

Preparation of this chapter was supported, in part, by grant number MDR 874-0400 from the National Science Foundation. All opinions expressed are those of the authors.

DESCRIPTION OF THE SECOND-GRADE PROJECT

The second-grade project, which began in 1985, grew out of several interview studies conducted with individual children in which we had documented some of the consequences of traditional textbook-based elementary school mathematics instruction (e.g., Cobb, 1986a, 1986b, 1987; Cobb & Wheatley, 1988). In concert with the findings of other researchers (e.g., Carpenter, Hiebert, & Moser, 1983; Erlwanger, 1973), we had concluded that many children came to view mathematics in school as an activity in which they were to follow procedural instructions to manipulate symbols, which did not need to signify anything beyond the marks themselves. Their mathematical activity in school typically seemed to be divorced from their out-of-school pragmatic problem-solving activity, and, as a consequence, the possibility of this activity being related to mathematics learned in school appeared to be the exception rather than the rule. It appeared that most students' primary goal in the classroom was not to understand mathematics in a meaningful way but to get work done by following specified procedures (Doyle, 1983; Marshall, 1988). In general, they seemed to have learned that mathematical interpretations, solutions, and answers were not the kinds of things that were open to discussion.

The potential contribution of telling further depressing stories of this type seemed somewhat limited. There was already a developing consensus in the mathematics education research community that the evidence was reasonably compelling and that something needed to be done about it. The most feasible approach at the time seemed to be to engage in what the NCTM Research Advisory Committee (1988) has since called transformational research by collaborating with teachers to explore what might be possible in the setting of public school instruction. In this regard, we were in full agreement with Tuthill and Ashton (1983) when they argued that

> without the assistance of practitioners whose classrooms are organized on the basis of a consistent paradigmatic perspective, educational scientists have no way of knowing whether their ideas are effective or not. (pp. 9-10)

As a first step, we conducted a yearlong classroom teaching experiment in collaboration with a second-grade teacher and, in the process, developed instructional activities by drawing on cognitive models of young children's arithmetical learning (e.g., Steffe, Cobb., & von Glasersfeld 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983).

The Second-Grade Classroom

The classroom was located near a midsize midwestern city in a school that served both a suburban and a rural population. This was a relatively large elementary school in which there were five additional second-grade classrooms. In the project classroom there were 20 students, the maximum number allowed under a state and federally funded program to improve primary education by reducing class size. The students were from a range of socioeconomic backgrounds, that is, from unemployed to professional families. In this school district, all children were randomly assigned to second-grade classes by the principal according to their reading levels in first-grade. The children's level of reading was used for classroom assignment because at this time the school district did not give a standardized achievement test to first graders.

The classroom teacher was at midcareer, having taught for 15 years, all in the second-grade. She had an undergraduate and a master's degree in elementary education from two of the state universities. She "volunteered" to be involved in the project after a request from her principal and after talking with us. She often humorously referred to this invitation when she speculated on the principal's rationale for choosing her for the project. It was her contention that he knew that language arts was her preference and "mathematics was not my thing," so therefore "working with a mathematics project might benefit me as well as the children."

The activities that were developed, along with the general instructional approach adopted, differed radically from those found in most elementary school classrooms in the United States. There was, for example, no individual paper-and-pencil seatwork or grading of written work. Instead, the children first attempted to complete the instructional activities with a partner and then participated in a teacher-orchestrated discussion of their mathematical problems, interpretations, and solutions. In general, both the materials and instructional strategies developed in the course of the classroom teaching experiment reflected the view that mathematical learning is a constructive, problem-solving process in which experience, activity, and communication are essential. This instructional approach has subsequently been found to be highly compatible with the curriculum reform advocated by the National Council of Teachers of Mathematics (1989) in their document on curriculum and evaluation standards.

OVERVIEW OF THE MONOGRAPH

In Chapter 1, we detail the manner in which we initially approached the study from a cognitive perspective by describing the rationale and methodology for the initial classroom teaching experiment we conducted during 1987–88. Chapter 2 centers on the evolution of the research to include a social interaction perspective. This section focuses on the development of the social norms that are crucial when establishing the classroom as a setting for meaningful mathematical learning. The process by which the teacher influenced her students' beliefs about their own roles, others' roles, and the general nature of mathematical activity as she initiated and guided the reconstruction of classroom social norms is described. In the course of their participation in this process, the students also contributed to the development of what might be called a problem-solving atmosphere in their classroom. In Chapter 3, we summarize our current theoretical orientation and methodology. In this section we discuss the theoretical constructs that evolved as we attempted to understand what might be going on in the classroom. Attention is given both to the problem of coordinating psychological and sociological analyses and to the rationale for the classroom teaching methodology. Chapters 4, 5, and 6 present analyses of classroom episodes in which opportunities for students to engage in mathematical activity arose once a problem-solving atmosphere had been established. These analyses of scenes from the classroom have two purposes. The first is pragmatic, that is, to offer "snapshots" that give the reader a sense for the way the members of the class interact as they work together. The second is theoretical, in that the episodes were selected to be illustrative of our theoretical constructs and methodological procedures as we attempted to make sense of the complexity of classroom life. It should be noted that these two sections, which deal with collaborative small-group work and teacher-orchestrated whole-class discussions respectively, are interrelated. On the one hand, children's mathematical activity as they interacted with their partners constituted the basis for the whole-class discussions. On the other hand, the children's realization that they would be expected to explain and justify their interpretations, solutions, and answers in the whole-class setting influenced the quality of their small-group work. In addition, the whole-class discussions served as settings in which certain mathematical meanings and practices but not others were legitimized, thus influencing subsequent small-group work.

As we have already noted, attempts to initiate and guide changes in educational practice at the classroom level occur within a wider sociopolitical setting constituted by administrators, elected officials, and parents as well as by teachers and researchers. Chapter 7 begins with a discussion of the theoretical and methodological considerations that guided the ethnographic analysis of the project within the wider community and then describes the multiple and, at times, conflicting perspectives that composed this wider setting. This aspect acknowledges that the process of reform in mathematics education is not limited to individual teachers and their students in classrooms but also extends to a broader level in which the community plays a significant role. Finally, the monograph concludes with a commentary, Chapters 8 and 9, by Martin Simon. Drawing on his own extensive experience of attempting to influence the teachers' activity during mathematics instruction, Simon identifies common themes and issues, discusses the possible contributions of the project to the field of mathematics education, and raises questions and concerns for reform in mathematics education.

Part 1

LEARNING MATHEMATICS: MULTIPLE PERSPECTIVES

Chapter 1

SECOND-GRADE CLASSROOM: PSYCHOLOGICAL PERSPECTIVE

Terry Wood
(An overview of work by Cobb, Wood, Yackel, and Wheatley)

Initially, constructivism, as a cognitive position, was the underlying theoretical and methodological perspective for our research in the second-grade classroom. This position, as a psychological view, has been articulated in mathematics education by von Glasersfeld (1984, 1987) and Cobb (1986a), in general, and more specifically by Confrey (1990), Labinowicz (1985), Steffe (1983), and Thompson (1985). As a theoretical perspective it is in direct contrast to the traditional view held that one learns by absorbing information that is transmitted by a more knowledgeable other or text. Instead, this position suggests that all knowledge is constructed and learners create knowledge for themselves by acting on the world (Piaget 1970, 1980). In addition to the tenet mentioned above, five principles formulate this perspective:

- Mathematical knowledge is actively constructed by the child.
- Children create new mathematical knowledge by reflecting on their physical and mental actions.
- Children's constructions undergo a process of continual revision.
- Children create their own individual interpretations of mathematics.
- Opportunities for learning occur during social interaction as children resolve conflicting points of view.

Further cognitive constructivism extends logically to a methodology that enables the researcher to investigate and describe students' cognitive structures and the development of their ongoing constructions. These methods include aspects of ethnography that entail observation, clinical interview, and microethographic analysis as ways to understand the processes by which individuals learn and adapt to their environment. It is from this psychological perspective that we began the yearlong classroom teaching experiment.

CLASSROOM TEACHING EXPERIMENT

The teaching experiment was derived from the Piagetian clinical interview procedure and extended to include teaching episodes by Steffe (1983). A teaching experiment is conducted in one-to-one settings in which the researcher acts

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as teacher. This enables the researcher, as teacher, to probe more extensively the processes by which individual children construct and develop mathematical knowledge. During the clinical interview, the researcher endeavors to infer the child's current ways of knowing about numerical processes. In the teaching episodes that follow, the researcher attempts to create situations for children to learn by posing tasks, asking questions, and making probes that create opportunities for children to question and reorganize their current ways of knowing. The researcher/teacher then tries to interpret the child's responses and ensuing mathematical activity to create new situations in which to continue to investigate children's learning. The clinical interviews, therefore, act as "bench marks" during the teaching episodes by providing baseline information about a child's understanding at various points in time. In Steffe's case, he used this methodology to identify the way in which individual children think. He then constructed a cognitive model of their thoughts that can be used to help interpret the thought patterns of other children. Thus, what children do or say in a situation, such as a classroom, can be understood in terms of how they view things (Cobb & Steffe, 1983).

The classroom teaching experiment was an attempt to extend the theoretical and experimental aspects of Steffe's research methodology to the pragmatic setting of the classroom to include 20 children in which the teacher, not the researcher, would provide instruction for the students. The intention was to create a situation that would, on the one hand, offer a more naturalistic setting for the researcher to study children's learning and, on the other hand, be compatible with the ongoing activities of the classroom. In a sense, it was an attempt to bridge the gap between research and practice. In order for this to occur, the researchers and the teacher assumed responsibility for certain aspects of the teaching experiment usually allocated to the investigator in the Steffe type of teaching experiment. It was initially anticipated that the teacher would conduct her lessons in a manner similar to that of the researcher in the teaching experiment, in that the teacher would develop an understanding of her students mathematical meanings as she was teaching. She would then use these ideas to create new situations that in turn might create opportunities for children to learn new mathematical constructions. This approach to instruction reflected our initial intention of creating settings in which individual children's thinking about mathematical problems was the central focus. The methodology was further modified in that the researchers assumed a major responsibility for conducting the clinical interviews and for posing the tasks, that is, the development of the problem-centered activities, for all 20 children rather than just 1. In the one-to-one teaching experiment, tasks were selected on the basis of the mathematical activity of an individual child, whereas in the classroom teaching experiment, activities were developed from analyses of all children's experiences and were designed to give rise to situations that were problematic for them. In order to develop these, the researchers were in the classroom on a daily basis during each mathematics lesson in order to make observations about the nature of the students' learning as they worked on the tasks. The teacher's role was to listen to her students as they expressed their thinking about the activities and to build from these thoughts by asking questions and making suggestions to create opportunities for their learning.

At the end of each week, a project meeting was held in which the researchers and teacher discussed the lessons of the week and determined the nature of the activities for the following week. Although this was done in consultation with the teacher, the major responsibility for the development and production of the daily activities remained with the researchers. We reasoned that she would thus not be burdened with the need to search for or develop instructional activities to meet her daily needs. This would allow the teacher time to develop her understanding of individual students' conceptions and incorporate these understandings about students into her teaching.

Instructional Activities

In attempting to instigate a classroom teaching experiment methodology, the selection and development of the classroom organization and instructional activities were grounded in a strong cognitive rationale and ensuing methodological considerations. The pedagogical advantages of these solutions of these selections were realized later as they were implemented in the classroom. The selections had to be compatible with implications from a constructivist theory, meet the methodological needs of the researchers, and yet fall within the constraints of the reality of the classroom. Accordingly, the instructional activities for the students emphasized problem-centered tasks and a classroom organization that consisted of pair-collaborative learning and class discussion.

Problem-centered activities. The constructivist view of learning as described by Steffe (1988) suggests:

> Mathematical learning is viewed as consisting in the adaptations children make in their functioning schemes as a result of their experiences to neutralize perturbations that can arise in one of several ways....Problem solving conceived of as goal directed activity is a crucial aspect of learning mathematical knowledge. (p. 5)

The implication for mathematics education is that the focus of instruction should be on problem-centered activities (Cobb, Wood, & Yackel, 1991a; Kamii, 1985; Thompson, 1985). By this, we mean that children should be attempting to resolve problematic situations that arise for them as they attempt to achieve their goals in the classroom. These personally experienced mathematical problems then create opportunities for them to learn in the classroom. Consequently, we drew on the cognitive models developed by Steffe and colleagues (1983, 1988) to develop the arithmetical instructional activities. We used the models to anticipate what might be problematic for children as they attempted to construct concepts of number and the operations for additive and multiplicative situations. The emphasis was on solving a few challenging problems rather than completing a large number of tasks as practice for the purpose of facilitating efficient computation.

As an example, the balance format (Figure 1.1) was used continuously throughout the year for a variety of arithmetical activities including basic facts, addition and subtraction, and multiplication and division. This particular example occurred at midyear and was part of a series of activities in various formats used to encourage children to develop a conceptual basis for ten. In this activity, children can solve these problems with a variety of solution methods ranging from modeling with manipulatives and counting by tens and ones to using efficient nonstandard algorithms. In addition to whole-number activities, other aspects ranged from spatial imaging and geometry to clock face reading, which were all presented in a problem-centered manner. A number of the activities developed reflect our agreement with the school administration to accept all the school district curriculum objectives for second-grade mathematics, regardless of whether we could justify these objectives in terms of cognitive models of students' mathematical learning.

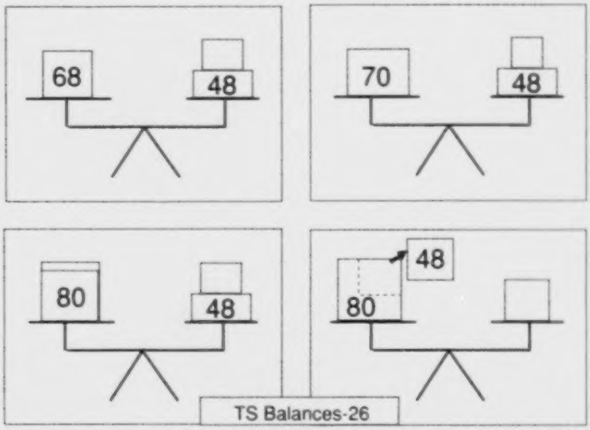

Figure 1.1. Example balance activity for thinking strategies and 2-digit addition and subtraction.

An example of the spatial activities that were employed is shown in Figure 1.2. These activities were presented on transparencies, which were shown on an overhead projector as a flashing activity in which children first drew what they saw and then discussed their images. The discussion frequently focused on the geometrical aspects of the figure. For example, the first design in Figure 1.2 elicited a discussion about the difference between a square and a rectangle, and the second drawing resulted in noticing vanishing points and infinity. Further information about the particulars of these spatial activities and possibilities for children's learning can be found in Yackel and Wheatley (1990).

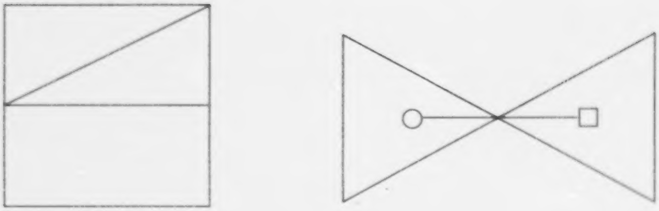

Figure 1.2. Two examples of spatial activities.

Methodologically, these activity sheets were records of students' mathematical activity and provided additional information about their current level of understanding. Further, these activities, being problematic in nature, were necessarily challenging for students, and to this end the classroom instructional strategies were essential.

Classroom Organization

Although the teaching experiment methodology and coinciding problem-centered activities were thought to be particularly well-suited for the purpose of investigating the processes by which individual children might construct numerical understanding, learning in the complexity of the classroom is a social process. These social aspects of learning cannot be neglected. Drawing on the research of Perret-Clermont (1980) and others (Doise & Mugny, 1979, 1984), we included pair collaboration and class discussion as additional aspects of the classroom instructional setting. By creating these situations, it was our intention to extend children's opportunities for learning to consider those settings in which social interaction could give rise to cognitive conflict for individuals. The situations that children find problematic take a variety of forms and can include resolving obstacles or contradictions that arise when they attempt to make sense of a situation in terms of their current concepts or procedures, accounting for a surprising outcome (for example, when two alternative procedures lead to the same result), verbalizing their mathematical thinking, resolving conflicting points of view, developing a framework that accommodates alternative solution methods, and formulating an explanation to clarify another child's solution attempt. Thus, we anticipated that genuine mathematical problems could arise in the course of social interactions as well as from an individual's attempts to complete an instructional activity. Accordingly, children typically worked in small groups on instructional activities for about 20 minutes, which was then followed by teacher-orchestrated whole-class discussion for another 20 minutes.

Pair collaboration. Pair collaboration allowed children an opportunity to work together in order to solve the instructional activities, which had a practical as well as theoretical aspect. Theoretically, the collaboration provided opportunities for children to participate in genuine communication as they engaged in mathematical activity. As they exchanged ideas and verbalized their mathematical thinking, opportunities arose for the resolution of conflicts (Barnes & Todd, 1977). Practically, collaborative settings created situations in which the teacher could learn about individual students' mathematical thinking as she listened, observed, and asked questions while interacting with the groups. Although the instructional activities encouraged children's active construction of concepts and operations, they also were intended to involve children in solving problems for which the procedure was not immediately known (Schroeder & Lester, 1989). In a traditional class, students working individually and encountering such problems would seek help from the teacher, creating the familiar "stack up" problem of a long line of children waiting for help. When children worked in pairs, however, they learned to rely on one another, eliminating students' many requests for help.

Whole-class discussions. The class discussions that followed the collaborative work enabled children to participate in a form of discourse in which the intent is to engage in genuine communication. For the researchers, methodologically, this presented an additional opportunity to listen to children explain their solutions. Further, the discussions provided the teacher with an opportunity to hear many of her students' ideas and thus provided more information for building cognitive models of children's thinking. These discussions were intended, from a theoretical perspective, to create another opportunity for students to learn as they reconstituted their solutions. Moreover, the discourse created situations for learning to occur as children listened to others' solutions and tried to understand alternative points of views.

In summary, each of the three aspects of the program was equally important and highly interrelated. The problem-centered activities provided opportunities for children to individually construct increasingly sophisticated concepts and operations, and the pair collaboration and class discussion created additional opportunities to learn as students engaged in social interaction. The problem-centered activities were designed to be challenging and open-ended to allow for a variety of solutions from the students. These activities, although open to multiple solutions, also emphasized the development of underlying conceptual operations. Given the nature of these activities, it was crucial that children engage in interactive situations in which discussion of their ways of thinking was encouraged and respected. The inclusion of pair collaboration provided opportunities for children to learn as they offered explanations to one another, listened to their partners' solutions, and came to a consensus. The class discussions were also necessary in that they created situations in which students were required to explain their answers and thus reflect as they listened to others' explanations. The opportunities that existed for children to learn in this classroom were critically related to the manner in which the three aspects functioned in a smooth fashion to create what Silver (1985) has called a "problem solving atmosphere."

METHODOLOGY

In light of the fact that the methodology used to investigate individual students' learning in the classroom was an extension of the theoretical perspective underlying the one-to-one teaching experiment of Steffe, the research falls in the qualitative tradition in which an interpretative stance guides the analyses (Atkinson, Delamont, & Hammersley, 1988; Erickson, 1986; Firestone, 1987; Jacob, 1987). In this approach the central issue for research is the meanings that humans give to their activity. From this perspective we acted as participant observers in the classroom, recording the events through videotapes and field notes. This allowed us to investigate the nature of classrooms as psychological, socially organized settings for learning and to examine the meaning perspective of the learners.

Classroom Data Collection

The data collection consisted of using video cameras to record each mathematics lesson for the entire school year. Two cameras were used to focus on four selected pairs of children during the small-group work. These eight children were videotaped in order to record in detail the children's mathematical activity. These video recordings, together with ethnographic field notes and copies of all the children's work, were collected for use in the analyses of the evolving nature of children's mathematical meanings and their social interaction in small groups. A single camera was used to record all whole-class discussions between the teacher and students during these daily lessons. Ethnographic field notes along with copies of the children's work were used to inform the analysis of the interaction and opportunities for learning that occurred between the teacher and the students.

The small-group videotapes for each pair were reviewed, and selected events were transcribed and notated for further analyses. A similar procedure was followed for videotapes of the whole-class discussions. These records and transcriptions of the videotapes along with the field notes were used to select an initial set of lessons for complete transcribing. These transcriptions then were used as the sources of episodes for detailed analyses (Erickson, 1986). The lessons were analyzed for the recurring patterns of interaction that occurred during the class discussions. This procedure is similar to that described by Edwards and Mercer (1987) and Wells (1985). Finally, the technique of microanalysis as used by Jungwirth (1991), Krummheuer (1988), and Voigt (1985, 1989) was used to describe in detail the nature of social interaction in the whole-class discussion. This procedure consists of line-by-line analysis of the discourse from selected episodes to describe the regularity of the process of interaction and the meaning that is conveyed between the participants. This type of analysis involves a process of inferencing based on the actions and discourse of the students and teacher.

Clinical Interviews

In addition to the extensive data collected during the classroom lessons, clinical interviews were conducted with the 20 children at the beginning, middle, and end of the year for the purpose of analyzing individual children's constructions of number and place-value concepts. These interviews provided information that aided our daily classroom observations of the students. The tasks used in the interviews were designed to investigate children's understanding of basic addition and subtraction facts, thinking strategies, and concept of ten. The use of algorithms for addition and subtraction in a variety of contexts (i.e., horizontal sentences with plastic numerals and vertical sentences on a worksheet, made up the remaining tasks. For further discussion and examples of the tasks see Cobb (1986b), and Cobb and Wheatley (1988).

Chapter 2

CREATING AN ENVIRONMENT FOR LEARNING MATHEMATICS: SOCIAL INTERACTION PERSPECTIVE

Terry Wood
(An overview of work by Cobb, Yackel, and Wood)

It was with the theoretical ideas about constructivists' view of learning discussed in the preceding chapter that we began our collaboration with the classroom teacher. We met weekly in the spring before the teaching experiment year to discuss our theoretical perspective and to view selected videotapes of interviews with her students in the fall (cf. Cobb, Yackel, & Wood, 1991; Cobb, Wood, & Yackel, 1990). Although we communicated our intentions in discussions about the importance of problem solving for learning and the necessity of social interaction and class discussion, it was still the teacher's obligation to enact these in the classroom. Admittedly we were well aware that children actively discussing challenging problems in primary grades was different from the way mathematics had been taught in the past, but we had not yet realized the extent to which these ideas would influence the practice of elementary school mathematics. These aspects—challenging problems, collaborative group work, and class discussion about students' solutions—were, for the teacher, against tradition. It was accepted practice for her to initiate grouped settings and discussions in social studies, science, and reading, but she did not do this in mathematics. It was against this background that the classroom teaching experiment began.

AN EXAMPLE OF A MATHEMATICS LESSON

Typically a class session began with the teacher leading a brief introduction intended to insure that the children understood what they would be working on for the day. Once the teacher was satisfied that the children understood the intent of the activities, she then passed out the activity sheets and small-group work began. Children worked in pairs on activities, which were on sheets of paper that provided room for students to write. Each pair received one sheet to share in completing the activity. Generally three to four sheets, each containing four to six problems, were available for the students to work on. Some children completed all the activity sheets, whereas others only finished one. The problem solving as pairs generally lasted 20 to 25 minutes and was followed by a class

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discussion of 15 to 20 minutes, after which the papers were collected by the teacher. During the small-group time the teacher initially moved about observing students as they worked. Once assured that all were working, she selected individual pairs to observe and listen to and intervened if necessary.

When it was time for class discussion, the teacher told the children to get ready because they would begin in one minute. The teacher began the discussion by writing a problem from one of the activity sheets on the overhead projector and then asked for volunteers to explain their solution to the problem. Pairs of children offered their explanations while the others listened. Following a pair's explanation, other children could ask questions for clarification or justification. Because the emphasis of the discussion was on students' solutions, on many occasions only a few problems were discussed.

On some days, the session might begin with a whole-class activity that would last 5 to 10 minutes. In this situation, the teacher would pose problems in activities such as "What's My Rule?" These sessions would be followed by the session as described above. And on a few occasions, the lesson would consist of a whole-class activity such as Spatial Imaging (cf. Yackel & Wheatley, 1990) that was led by the teacher.

THE DEVELOPMENT OF SOCIAL NORMS

The expectations for children's actions in the mathematics class were quite different from their previous experiences in school and in this classroom in other subjects (Wood, Cobb, & Yackel, 1990). In those situations, the students were expected to learn what the teacher wanted them to know rather than express their own thoughts (Edwards & Mercer, 1987; Weber, 1986). However, in this mathematics class it was necessary for children to express their thinking in order to create opportunities for learning and so that their existing constructions could be investigated by both the teacher and researchers.

Additionally, a cognitive constructivist perspective on learning necessarily implies ways of teaching in which children are acknowledged as active constructors of knowledge. These implications for teaching are as follows:

- Teachers should provide students with instructional activities that will give rise to problematic situations.
- Children's actions, which are logical to them but may be irrational from an adult perspective, should be viewed as rational by the teacher.
- Teachers should recognize that what seem like errors and confusions, are children's expressions of their current understandings.
- Teachers should realize that substantive learning occurs in periods of conflict, confusion, surprise, over long periods of time, and during social interaction.

These implications were not provided for the teacher as dictates of what she should do but were the logical outcome of earlier discussions about children's learning (Cobb, Wood, & Yackel, 1990).

Using these premises of children's learning as her guideline, the teacher initiated the mutual construction of a different set of norms for mathematics lessons as she acted to help the students reconceptualize their role during mathematics instruction. Her intention was for the children to figure things out for themselves and to express their ideas in the public arena of whole-class discussions. Additionally, during small-group work she expected them to cooperate and work together to solve problems and to agree on an answer. Her expectation that the children would express their thoughts placed the students under the obligation of having to recall their solutions and explain them to others during the whole-class discussion.

In small groups, the children were obligated to exchange ideas and solution methods and ultimately to agree on an answer (Yackel, Cobb, & Wood, 1991).

Conversely, if the teacher had expectations for the students, then she had to reciprocate and accept certain obligations for her actions as well. From the children's point of view a certain amount of risk was involved in fulfilling the teacher's expectations. A child thinking about how to solve a problem privately is one thing, but it is quite another matter to express those thoughts in a public setting. A child's thinking in this situation can be publicly scrutinized and evaluated not only by the teacher but also by his or her peers.

Norms for Class Discussion

The nature of the teacher and student interaction that occurred within the whole-class discussion was crucial to establishing the social norms that were necessary for developing a setting in which the children would feel psychologically safe to express their mathematical thinking.[1] The teacher's intention as she led class discussion was to encourage children to verbalize their solution attempts. Situations like this give rise to possibilities for learning as students attempt to reconstruct their solutions (Levina, 1981) and resolve conflicting points of view (Perret-Clermont, 1980). The manner in which the mutual construction of the different norms for class discussion evolved is reflected in the early classroom interaction patterns, in which two levels of discourse occurred. At one level, they talked about doing mathematics, whereas at the other, they talked about talking about mathematics. On the occasions when they talked about how they were to talk about and do math, the teacher typically initiated and attempted to control the conversation. When they talked about mathematics, she acted to orchestrate and guide the discourse. The framework of the social norms created a setting for students in which their thinking was respected and they could say what they really thought. The following episode, which occurred during the first week of school, illustrates how they mutually constructed the social context in which students could express their mathematical thinking.

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[1] We have previously written fairly extensively about the development of the social norms and their importance in establishing an environment for children to learn mathematics meaningfully. Rather than repeat this work in this text, if the reader is interested in further details, the following articles may be of interest: Cobb, Wood, and Yackel (1990); Cobb, Wood, and Yackel (1991a); Cobb, Yackel, and Wood (1989); Cobb, Yackel, Wood, Wheatley, & Merkel (1988); Wood (1989); Wood & Yackel (1990); Wood, Cobb, and Yackel (1991); Yackel (1989); Yackel, Cobb, & Wood (1991); Yackel, Cobb, Wood, Wheatley, & Merkel (1990).

The students were discussing a word problem that was shown on the overhead projector.

Teacher: Take a look at this problem. The clown is first in line. Which animal is fourth? Peter.

Peter: The tiger.

Teacher: How did you decide the tiger? ... Would you show us how you got the fourth?

Peter: (Goes to the front of the room and points to the screen). I saw the clown and then ... (He counts the animals to himself). Oh, the dog [is fourth]. (He hesitates.) Well, I couldn't see from my seat. (He looks down at the floor.)

Teacher: Okay. What did you come up with?

Peter: I didn't see it. (He goes back to his seat).

At this point, the teacher realized that although her intention was for him to give his solution, she had put him in the position of having to admit his answer was wrong in front of the entire class. Then the teacher initiated explicit comments.

Teacher: That's okay, Peter. It's all right. Boys and girls, even if your answer is not correct, I am most interested in having you think. That's the important part. We are not always going to get answers right, but we want to try.

Her comments were focused on talking about how in this class they were going to talk about mathematics. In this example she told the students that thinking was valued even more than right answers. These mutual obligations and expectations were negotiated and renegotiated by the teacher and students as they established an interaction pattern that would form the basis for their activity. These mutually constituted patterns of interaction were taken for granted and made possible the smooth functioning of their collective activity.

In addition to expecting students to express their thinking, they were expected to explain and justify their solutions and listen to others' explanations. Because the class discussion followed pair collaboration, the children were obligated to be prepared to give their solutions to the problems when called on. Being unprepared meant the rest of the class would wait while they redid the problem. Often this resulted in students losing interest in the discussion. As an example, in the following episode Barbara and Steve have come to the front to explain their solution to a problem involving adding the two-digit numbers 29 + 19 given in the context of a word problem. The teacher had asked Barbara to explain how she got her answer of 48. The pair began by writing on the board but providing no explanation. After a pause the teacher intervened:

Teacher: You two work on it. ... You whisper and figure it out for yourselves. Obviously, you don't have it all quite worked out well enough to explain it to us. You got one answer but you're not sure how you did it.

She turned to address the class as she made the last statement and called on another pair. In this way, she has made it very clear that when students explain their answer, they must have agreed on the answer and solution in their small group before attempting to explain it to the whole class.

Norms for Pair Collaboration

The obligations and expectations mutually constructed during whole-class discussion also provided a framework for their activity as they worked in small groups. In this situation they worked to solve problems, to agree on an answer, and to respect each others' ideas. In addition, there were norms for individual activity, including that the children figure out solutions that were meaningful to them, that they explain their solution methods to their partners, and that they try to make sense of their solving attempts. The teacher used the setting of the class discussion to explicitly discuss the nature of the students' obligations. The following example illustrates the directive manner in which she conducted these discussions. She began the lesson by asking, "What are you going to be doing as you are working together?"

Lisa: Solve the answers.

Teacher: Find the solution. What else is your responsibility to you and your partner ...to each other?

Ron: Share.

Teacher: Share. What else, Adam?

Dan: Cooperate.

Teacher: Well, what if ... another responsibility?

Children: To agree.

Teacher: Not necessarily agree, but what else?

Katie: Well, if Ron and Charles are working together, if Charles got the answer, he's supposed to tell Ron how he got that answer.

The mutual construction of classroom norms that has been described was crucial to establishing the setting for learning. Furthermore, in this classroom, because the children's intention was to engage in meaningful mathematical activity with one another as they completed instructional activities (Yackel, Cobb, Wood, Wheatley, & Merkel, 1990), there was no need for an external reward system to motivate the children's activity. In the terms of Nicholls (1983), they were task-oriented as opposed to ego-oriented and have developed what Kamii (1985) calls autonomy. Thus, children in the classroom developed both social autonomy, taking responsibility for their conduct, and intellectual autonomy, taking responsibility for their own learning (cf. Cobb, Yackel, & Wood, 1989).

SUMMARY

It became evident that a psychological perspective alone could not account for the complexity of the events occurring in the classroom. Establishing social norms that provided the setting in which children engaged in meaningful activity was an aspect of social interaction not considered prior to the classroom teaching experiment. As these norms became accepted, the students participated in a type of discourse in which they were expected to explain and justify their solutions and listen to others. The teacher acted to initiate and guide students' learning by posing questions and highlighting children's expectations. As students engaged in this discourse, their personal meanings were negotiated until an agreement was reached. The establishment of taken-as-shared meanings between the participants enabled mathematical ideas to be established by members of the class.

Chapter 3

THEORETICAL ORIENTATION

Paul Cobb, Erna Yackel, and Terry Wood

Thus far, we have chronicled the evolution of our initial cognitive constructivist theoretical and methodological premises and the ensuing transformation to include a social interactionist perspective that derived from a similar philosophical viewpoint. In this chapter, we bring together both the psychological and sociological positions that are exemplified to discuss our current theoretical orientation, *social constructivism*, and consequent constructs that will be exemplified in Part 4 and Part 5.

We initially viewed learning primarily from a psychological perspective in which individual children construct their mathematical ways of knowing by reorganizing their mathematical experiences in order to resolve their problems as they attempt to achieve their goals. Our work was guided by a radical constructivist theory of knowledge (von Glasersfeld, 1984, 1987) that rejects the transmission-of-knowledge model of learning and teaching that has typified traditional mathematics instruction. This traditionally held view of mathematics as a preexisting body of knowledge to be transmitted to the student from an outside source is eliminated in favor of a constructivist perspective which posits that students construct their own knowledge on the basis of their experiences and activity. According to von Glasersfeld (1987), "to have 'learned' means to have drawn conclusions from experience and to act accordingly" (p. 8). Learning occurs when individuals reflect on their activity, including sensory motor and conceptual activity, and reorganize their interpretive framework. Our work was also influenced by cognitive models of young children's arithmetical learning (Steffe et al., 1983) and their development of the concept of ten and nonstandard algorithms (Cobb & Wheatley; 1988, Steffe et al., 1988). Thus, we viewed mathematical learning as an active problem-solving process in which children reorganize their mathematical ways of knowing to resolve situations that, in their interpretation, give rise to obstacles or contradictions. This view was applied to all mathematical learning including the construction of arithmetical algorithms and was not restricted to instructional situations that are typically labeled "problem solving." It should also be noted that, from this perspective, mathematical problems cannot be given to students readymade. Instead, the sole basis for children's mathematical constructions consists of their personal experiences as they actively interpret situations while attempting to achieve their purpose in the classroom. The approach we took

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to the development of instructional activities and to the classroom organization were influenced by this view of mathematical learning as a problem-solving process. The cognitive models were used to anticipate the situations that might give rise to mathematical problems for children at different conceptual levels as well as the constructions they might make as they attempted to resolve their problems. In addition to this focus on problems that might arise from children's individual task interpretations, we also considered that contradiction and conflict could arise for children in the course of their social interactions in the classroom. Our position at that time was compatible with the work of several neo-Piagetian researchers who had investigated how social interaction can serve as a catalyst for cognitive development (e.g., Doise & Mugny, 1979, 1984; Doise, Mugny & Perret-Clermont, 1975; Perret-Clermont, 1980). As part of their rationale, they noted that Piaget had stressed in his early writings that social interaction was necessary for the development of logic, reflexivity, and self-awareness. Their accounts of an original series of studies described how conflicts between learners' interpretations in the course of social interactions can give rise to individual cognitive conflicts. In the process of resolving their cognitive conflicts, children can re-present and reflectively abstract from their activity, thus reorganizing their mathematical ways of knowing. The implications we drew from this work were that children would benefit from working collaboratively on mathematical tasks and that they should be encouraged to discuss their mathematical interpretations, solutions, and answers with others. As a consequence, the general instructional approach used in the classroom involved small-group work followed by whole-class discussions. Given this theoretical orientation, it initially seemed reasonable to make individual students' construction of mathematical knowledge the primary focus of our investigation. Within the first few days of our involvement in the classroom, we began to revise this initial emphasis and to reconsider some of our basic assumptions about children's mathematical learning in light of the complexity of classroom life.

THE SOCIAL REALITY OF MATHEMATICS CLASSROOMS

At our suggestion, the teacher encouraged her students to verbalize their mathematical interpretations and solutions during whole-class discussions at the very beginning of the school year. Our intent in making this suggestion was twofold. On the one hand, we had anticipated that various learning opportunities would arise for the students as they attempted to reconstruct their solutions, understand alternative solutions, and resolve conflicts between incompatible interpretations (Levina, 1981; Perret-Clermont, 1980; Sigel, 1981). On the other hand, we intended that the discussions would provide the teacher with opportunities to listen to students' explanations. In that way, she might begin to appreciate that students' mathematical activity can be rational for them, given their understandings and purposes, regardless of how bizarre it might seem initially from her adult point of view. She would then have reason to engage her students in what Rommetveit (1985) called genuine communication. As Uhlenbeck (1978) noted, communication that involves the explicit negotiation rather than the imposition of meanings only occurs if

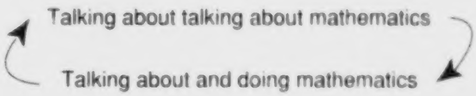
> the hearer always takes the view that what the speaker is saying somehow makes sense. It is this certitude which makes him try to infer ... what the speaker actually is conveying to him.... It is difficult to exaggerate the importance of this very general attitude. (p. 190)

As with many intended plans, our attempt to support the evolution of communicative discourse in the classroom immediately encountered an unanticipated difficulty. The teacher's and our expectation that the children would verbalize how they had actually interpreted and attempted to solve instructional activities clashed with the expectations they had developed on the basis of their prior experiences with class discussions in school. As Mehan's (1979) analysis indicates, such discussions are typified by an interactional sequence in which the teacher asks a question, a student gives a response, and the teacher evaluates the response. In general, students are typically steered or funneled toward an officially sanctioned solution method or answer during these dialogues (Bauersfeld, 1980; Voigt, 1985). The second graders in the project classroom therefore assumed that they were obliged to try and infer the response the teacher had in mind rather than to articulate their own understandings. As a consequence of these previous experiences, the teacher had to initiate and guide the renegotiation of classroom social norms, thereby influencing her students' beliefs about their own roles, the teacher's role, and the general nature of mathematical activity (Cobb et al., 1989; Yackel et al., 1991).

We were able to distinguish between two levels of conversation in the whole-class discussions when we began to analyze this process. At one level, the teacher and students did and talked about mathematics, whereas at the other level they talked about talking about mathematics. In general, the teacher and students explicitly negotiated mathematical meanings when they did and talked about mathematics. At this level, the teacher expressed her authority in action by helping students express their thinking, framing conflicting solutions and answers as problems to be resolved, subtly highlighting and legitimizing selected aspects of the students' contributions that were potentially productive with regard to their subsequent learning, and so on. Nonetheless, there was a relative symmetry between the teacher's and students' roles and a mutual respect for others' mathematical thinking at this level of conversation. In particular, the teacher did not attempt to steer the students to a predetermined mathematical solution or answer, but instead capitalized on their constructive activities to arrive at a consensus. Further, the students could take the initiative to explain an insight, to challenge another's interpretation, or to raise a question.

In the course of discussions in which the teacher and students did and talked about mathematics, situations arose where their expectations for each other were in conflict. From the teacher's point of view, the student had failed to fulfill an obligation, thereby transgressing a social norm. For example, a student might fail to explain his or her solution when the teacher considered that the situation warranted an explanation. The teacher typically attempted to resolve the conflict by framing the incident that arose as they did and talked about mathematics as a paradigm case in which to discuss her expectations with the students. In the process, the conflict became an explicit topic of conversation, and the teacher and students began to talk about talking about mathematics. At this level of conversation, the teacher was much more directive as she discussed with her students how they ought to interpret this situation and similar ones. It follows from Bateson's (1973) analysis of context that two levels of conversation are logically distinct though interdependent. Consequently, it makes sense to say that the teacher overtly exercised her institutionalized authority to make it possible for the students to participate in dialogues in which they said what they really thought mathematically, thus creating opportunities to learn.

In general, the relationship between the two levels of conversation was reflexive in nature. On the one hand, incidents that arose when doing and talking about mathematics were framed as paradigmatic situations in which to talk about mutual obligations and expectations. Thus, the topics discussed when talking about talking about mathematics emerged as the teacher and students did and talked about mathematics. On the other hand, the explicit negotiation of social norms that regulated classroom social life influenced the teacher's and students' subsequent activity as they did and talked about mathematics. This reflexive relationship can be represented as follows:


Talking about talking about mathematics
Talking about and doing mathematics

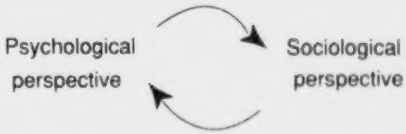
In practice, we attempted to identify classroom social norms by analyzing the evolving patterns and regularities that occur in classroom social interactions (Cobb, Wood, & Yackel, in press; Cobb et al., 1988). These patterns are, for the most part, outside the conscious awareness of both the teacher and the students and are repeatedly reconstructed in the course of interactions (Bauersfeld, 1980; Voigt, 1989). In other words, the patterns consist of coordinated sequences of routine individual actions, and at each occurrence, the development of a pattern begins anew—the enactment and the construction of the pattern are synonymous. Thus, although the teacher and students do not have a "blueprint" of the interaction pattern, each knows how to act appropriately in particular situations. It is these patterns in social interactions that reveal the largely implicit social norms negotiated by the teacher and students, the norms that constitute the social reality within which they teach and learn mathematics.

Implicit and Explicit Negotiation

As we have seen, the teacher and students explicitly negotiated their obligations and expectations when they talked about talking about mathematics. *Explicit negotiations* of this sort that occurred when differences between the teacher's and students' interpretations became topics of conversation can be contrasted with *implicit negotiations*. Both symbolic interactionists (e.g., Blumer, 1969; Mead, 1934) and ethnomethodologists (e.g., Mehan, 1979; Mehan & Wood, 1975) have noted that the participants in any social interaction continually orient and adjust to each others' activity. These theorists therefore view social interactions as processes that necessarily involve the negotiation of meaning. This usage of the term negotiation is not synonymous with its typical usage in the mathematics education research community, because for symbolic interactionists, negotiation does not imply that the participants are consciously aware of differences in their individual interpretations. Further, the participants may not realize that their personal interpretations have evolved in the course of the interaction. One of the symbolic interactionists' primary concerns is in fact to understand the process by which these largely unnoticed shifts and slides of meaning occur. In the teaching experiment classroom, for example, the teacher and students continually adjusted their obligations and expectations in the process of talking about and doing mathematics (Cobb et al., in press). We consider that this is an instance of implicit negotiation, because their obligations and expectations were not an explicit topic of conversation and because the adjustments they made were for the most part outside their awareness as they discussed their mathematical interpretations and solutions. It was only when this implicit process of negotiation of meaning broke down that they became aware of differences in their obligations and expectations and explicitly negotiated their roles and responsibilities by talking about talking about mathematics. This process of explicit negotiation corresponds to Bishop's (1985) use of the term when he contrasted teaching by negotiation with teaching by imposition. In the most general terms, implicit and explicit negotiation are the key processes by which we account for the teacher's and students' construction of consensual meanings and thus their attainment of at least temporary states of intersubjectivity.

Cognitive and Sociological Perspectives

Thus far, we have focused on classroom social norms but have said little about individual children's beliefs. In doing so, we have approached classroom social life from a sociological rather than a cognitive perspective (Cobb, 1989). Following Comaroff (1982) and Lave (1988), we take the relationship between individual experiences and a mutually constructed social reality to be reflexive. This can be represented as follows:



We can illustrate this relationship by considering classroom social norms and what we take to be their cognitive correlates, the teacher's and children's individual beliefs about their own roles, others' roles, and the general nature of mathematical activity (Cobb et al., 1989). In the discussion of social norms, we discussed regularities or patterns in classroom social interactions that, from the observer's perspective, constitute the grammar of classroom life. In doing so, we tacitly assumed that these regularities are manifestations of shared knowledge (Gergen, 1985). For example, if we repeatedly observed that students routinely complained to the teacher when someone told them an answer, we might infer that they share a belief in the value of developing their own mathematical understanding. However, if we were to view interactions from the cognitive perspective and focus on individual children's interpretations of their own and others' activity, discrepancies in their individual beliefs might well become apparent. This suggests that the most that can be said, when interactions proceed smoothly, is that the teacher's and students' beliefs fit so that each acts in accordance with the other's expectations (Bauersfeld, 1988; Schutz, 1962; Voigt, 1989; von Glasersfeld, 1984). We will therefore speak of their beliefs as being *taken-as-shared*[1] to indicate that the experience of intersubjectivity does not require a correspondence or match between individual interpretations (Streeck, 1979). Instead, it is only necessary that the teacher's and students' interpretations of social situations are compatible for the purposes at hand. Situations in which social norms are explicitly negotiated occur when a lack of fit becomes apparent—when either the teacher's or a student's expectations are not fulfilled. In sociological terms, explicit negotiations occur when there is a perceived breach of a social norm (Much & Shweder, 1978). It was by capitalizing on such breaches and making them topics of conversation that the teacher and, to an increasing extent, the students initiated and guided the explicit negotiation of social norms and thus influenced others' beliefs. These beliefs in turn constituted the cognitive bases for their individual interpretations of the situations that arose in the course of social interactions. We represent the relationship between beliefs and social norms as follows:

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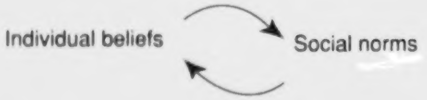
[1] The term originated with Schutz (1962) as *taken-to-be-shared* and was later changed to *taken-as-shared* (Streeck, 1979). It refers to the meaning that is thought to be shared by others. Although meanings that individuals hold are unique to them, the ability to communicate with others depends on the negotiation of meaning that is thought to be shared by others. "'Shared agreement' refers to various social methods for accomplishing the member's recognition that something was said-according-to-a-rule and not the demonstrable matching of substantive matters. The appropriate image of common understanding is therefore an operation rather than a common intersection of overlapping sets." (Garfinkel (1967) p. 30). These taken-as-shared meanings are held between the members of the group and are identified from the observer's perspective.



This relationship can be summarized by saying that individual interpretations that fit together constitute the social norms that constrain the individual interpretations that generate them. This, we believe, is an instantiation of the general relationship between the communal culture of social reality and individual experience.

We previously suggested that social norms can be identified by analyzing patterns in social interactions. Beliefs, as the cognitive correlates of social norms, constitute the largely tacit knowledge in terms of which students interpret social situations. These beliefs can be inferred by analyzing the obligations that individual children accept for their own activity and the expectations that they have for others' activity in specific situations. As a consequence, a student's inferred beliefs about his or her own role in the classroom, others' roles, and the general nature of mathematical activity can be thought of as a summarization of the obligations and expectations attributed to the student across a variety of situations. For example, if we observed that a student repeatedly became irritated or disgruntled when the teacher offered assistance by asking questions to encourage reflection, we would have some indication that the student expected the teacher to state explicit instructions on how to complete the task. This in turn might indicate that the student believed his or her own role was to follow procedural instructions. As this example makes clear, we are interested in students' beliefs-in-action, the largely unarticulated beliefs that underpin their activity in the classroom. In this regard, our treatment of beliefs is consistent with that of the pragmatists Dewey, James, and Pierce. In their view, a "willingness to act and ... the assumption of some risk and responsibility for action in relation to a belief represent essential indices of actual believing" (Smith, 1978, p. 24).

The complementarity between individual students' beliefs and classroom social norms becomes apparent when we note that an analysis of social norms specifies the evolving social reality or classroom mathematics tradition within which students construct their beliefs. Conversely, an analysis of students' beliefs documents how they reorganize their mathematical world views or, as phenomenologists would say, their natural attitudes in the classroom as they participate in the mutual construction of the classroom mathematics tradition. As a consequence, our focus on social norms does not deny that students interpret classroom situations in terms of their individual beliefs and values. Instead, we are suggesting that they construct these interpretive schemes as they attempt to be effective in the classroom and, in the process, contribute to the development of a particular classroom mathematics tradition. In this regard, social norms can be thought of as taken-as-shared beliefs that constitute a basis for communication and make possible the relatively smooth flow of classroom life.

SCHOOL MATHEMATICS AND INQUIRY MATHEMATICS

We introduced the notion of the classroom mathematics tradition when discussing the implicit and explicit negotiations of social norms that occurred as the teacher and students talked about talking about mathematics. This notion of the classroom mathematics tradition is closely related to a variety of theoretical constructs developed by other researchers who have analyzed the social reality of mathematics classrooms. These constructs include the classroom discursive practice (Walkerdine, 1988), the tradition of classroom practice (Solomon, 1989), and the classroom subculture or microculture (Bauersfeld, Krummheuer, & Voigt, 1988). In each case, the researchers developed their constructs by focusing on the development of taken-as-shared, or normative, suppositions, assumptions, and interpretations that make communication possible. It is these often implicit taken-as-shared social and mathematical understandings that constitute what we call the classroom mathematics tradition.

As the term implies, a classroom mathematics tradition is in many ways analogous to a scientific research tradition (Cobb, Wood, & Yackel, 1991b). For our purposes, it suffices to note that both are created by a community and that both influence individuals' construction of scientific or mathematical knowledge by constraining what can count as a problem, a solution, an explanation, and a justification (Barnes, 1982; Knorr-Cetina, 1982; Lampert, 1988; Tymoczko, 1986). It is our contention that the teacher who collaborated with us during the teaching experiment initiated and guided a transformation of the mathematics tradition in her classroom. As we have noted, her students learned what was expected of them in the setting of traditional school mathematics instruction as a consequence of their experiences while interacting with their first-grade teachers. In explicitly negotiating social norms with her students, the project teacher initiated them into an interpretive stance in which mathematical interpretations and solutions were explainable, justifiable, and, more generally, eminently discussable. By doing so, she, in effect, initiated and guided the analogue of what Kuhn (1970) called a scientific revolution in her classroom. Following Richards (1991), we call the mathematics traditions typically established during textbook instruction the *school mathematics tradition*, and use the *inquiry tradition* to describe project classrooms.

Mathematical Practices

By analogy with our suggestions that classroom social norms can be viewed as the sociological correlates of individual students' beliefs, the mathematical practices institutionalized in the classroom can be thought of as the sociological correlates of individual students' mathematical ways of knowing. In proposing this analogy, we are contending that mathematics does not consist of timeless, ahistorical facts, rules, or structures but is continually negotiated and institutionalized by a community of knowers. This contention is consistent with recent developments in the philosophy of mathematics (Lakatos, 1976; Tymoczko, 1986), with historical analyses that demonstrate that mathematical practices evolve (Bloor, 1983; Grabiner, 1986; Kitcher, 1983), and with the empirical finding that mathematical practices differ across communities (Carraher, Carraher, & Schliemann, 1985; D'Ambrosio, 1985; Saxe, 1991). In this view, both individual mathematical activity and mathematical communication in the classroom occur against a background of mathematical practices that have been institutionalized by the classroom community and are taken as self-evident by its members. Thus, the background for mathematical activity and discourse consists of the current, potentially revisable results of prior implicit and explicit negotiations of mathematical meanings as well as of social norms. Given that this background of taken-as-shared mathematical meanings and practices emerges and is continually modified in the course of social interactions, we call it the interactively constituted basis for mathematical activity.

It is with this discussion of taken-as-shared mathematical meanings and practices that the modifications we have made to our initial view of the nature of mathematical learning become apparent. We initially took an almost exclusively cognitive perspective and argued that students learn as they reorganize their experiences to resolve what they find problematic. The limitations of our initial position come to the fore once we consider the collective as well as the individual nature of mathematical activity and note with Solomon (1989) that the mathematical problems students attempt to solve and the solutions they construct in the course of their development have a social as well as a cognitive aspect. As a simple illustration, consider a situation in which two young children each count to solve an arithmetical task as they work together. Clearly, the children did not spontaneously develop the ability to count entirely on their own but were initiated into this mathematical practice by others who were already engaged in it. It other words, the constructions they made to resolve previously encountered problems were precisely those that made it possible for them to participate in the taken-as-shared mathematical practices of wider society. It is in this sense that their resolutions to prior problems have a social aspect and that learning is a process of acculturation as well as of individual construction. This is one half of the story, and if we said no more we would implicitly adopt a behaviorist stance towards mathematical activity. The other half of the story emphasized the active nature of the children's constructions as they attempt to achieve their goals while interacting with a parent or, perhaps, a more capable child. In particular, the children have each constructed both the conventional number word sequence and conceptual operations that make it possible for them to create increasingly sophisticated types of units to count (Steffe et al., 1983). In doing so, they participated in the interactive constitution of a taken-as-shared basis for mathematical activity with more mature members of society. The conceptual constructions they actively made were therefore social through and through.

Now suppose that the two children produced different answers when they each counted to solve the task. It is self-evident to an acculturated member of society that two separate counts should give the same result and that a mistake has been made if two children get two different results. The two children would also have to understand this apparently natural aspect of counting in order to interpret the situation in which they arrived at differing answers as an *interpersonal conflict* that needed to be resolved. That is, they would have to make interpretations that were taken-as-shared with mature members of society in order to construe the situation as an interpersonal conflict. However, if their activity is to be motivated by more than the blind adherence to a meaningless principle of counting, then it must be accounted for in cognitive terms by constructions they had previously made while interacting with more capable others. At a minimum, the experiential result of counting for each child would have to be a counted collection that each took-as-shared with the other (cf. Steffe et. al., 1983).

As this example illustrated, what is construed as a conflict or a problem and as a conceptual advance has a social aspect even at the most elementary levels of mathematics. In effect, mathematical problems arise for students against the background of the taken-as-shared ways of knowing that they actively constructed in the course of their prior mathematical acculturation. To the extent that they make conceptual advances while resolving these problems, they elaborate their taken-as-shared mathematical understandings, and these constitute the background against which further problems arise.

As a further illustration of the social nature of mathematical activity, it is readily apparent that we, as members of the wider community, take it for granted that whole numbers are composed of units of ten and of one. We, as members of this community, have jointly institutionalized this conceptual practice as a mathematical truth (cf. Wittgenstein, 1964). However, many second graders do not yet make this construction, and consequently teachers have to initiate and act to guide the institutionalization of this practice in their classrooms so that it will eventually be experienced as a mathematical truth by their students. In doing so, the teachers are, from the cognitive perspective, facilitating individual students' construction of mathematical knowledge. In the classroom we observed, the interpretation of numbers in terms of units of ten and one became increasingly taken-as-shared in that it was beyond need of justification. However, an analysis of individual students' mathematical conceptions indicated that this mathematical practice had a variety of qualitatively distinct meanings for them. From the cognitive perspective, their differing conceptions of place-value numeration were compatible in that they could talk about mathematics for the purposes at hand without becoming aware of discrepancies in their mathematical interpretations. In short, they individually constructed conceptualizations that were adequate as they completed tasks and engaged in collaborative mathematical activity with others.

The relationship between individual students' mathematical ways of knowing and the mathematical practices institutionalized by the classroom community is an instantiation of the relationship between the cognitive and sociological perspectives as shown in Figure 3.1. This relationship can be summarized by saying that individual mathematical activities that fit together constitute the institutionalized mathematical practices that constrain the mathematical activities that generate them. This conception of the relationship between individual and social phenomena emphasizes the central role that the explicit negotiation of mathematical meanings plays in the inquiry mathematics tradition. As Bruner (1986) put it:

> It follows from this view of culture as a forum that induction into the culture through education, if it is to prepare the young for life as lived, should also partake of the spirit of a forum, of negotiation, of the recreating of meaning. But this conclusion runs counter to traditions of pedagogy that derive from another time, another interpretation of culture, another conception of authority—one that looked at the process of education as a transmission of knowledge and values by those who knew more to those who knew less and knew it less expertly. (p. 123)

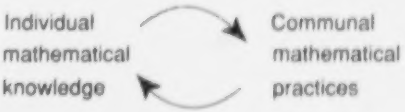


Figure 3.1 The relationship between psychological and sociological.

SUMMARY AND DISCUSSION

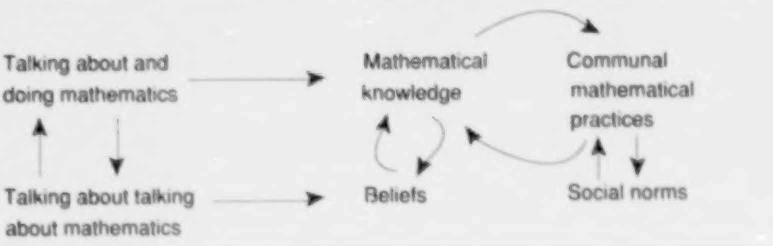
Two Perspectives

The various reflexive relationships we have discussed in this chapter are shown in Table 3.1. The relationships between individual mathematical knowledge and individual beliefs and between communal mathematical practices and communal social norms are analogous to the relationship between a scientific community's creation of scientific knowledge and its creation of a scientific research tradition. This scheme is the result of our attempts to make sense of life in the classroom and summarizes the general way in which we now look at the process of learning and teaching mathematics. We would acknowledge that it is incomplete in that it ignores the broader sociopolitical setting and the function of the school as a societal institution. To address this issue, we would have to explicate the relationship between microsociological and macrosociological processes. In this regard, Soviet activity theory (e.g., Leont'ev, 1978; Minick, 1987) might well prove to be of value.

Complementarity

The various competing and sometimes incommensurable paradigms that guide American mathematics education research have historically focused on the mathematical cognitions of individual students. Some of these paradigms have evolved in recent years to include a consideration of students' beliefs. However,

Table 3.1
Partial Relationships Between Observer's Perspective and Levels of Discourse



Levels of Discourse	Perspective Psychological	 Sociological
Talking about and doing mathematics	Mathematical knowledge	Communal mathematical practices
Talking about talking about mathematics	Beliefs	Social norms

with rare exceptions (e.g., Lampert, 1990; Schoenfeld, 1987), American mathematics educators have yet to take a sociological stance to the learning-teaching process. We have learned in the course of our research endeavor that it is essential to adopt this perspective even if one's primary concern is to understand individual students' construction of mathematical knowledge. Students' mathematical learning is influenced by both the mathematical practices and the social norms implicitly and explicitly negotiated and institutionalized by the classroom community. These norms and practices in effect constitute the immediate social situation of students' mathematical development. Conversely, we have argued that students actively participate in the interactive constitution of this evolving social situation and that consequently the forms it can take are constrained by students' current cognitive constructions. In short, the teacher's and students' cognitions were constituted and were constrained by their ongoing face-to-face interactions, and these interactions in turn constituted and were constrained by the classroom mathematics tradition viewed as an evolving social practice.

More generally, the position we have come to in the course of our work finds value in both cognitive and sociological interpretations of mathematical activity. As a consequence, the question of whether mathematics is essentially cognitive or whether it is essentially sociological or cultural in nature is considered to be irrelevant. We have instead suggested that it is useful to see mathematics as both a cognitive activity constrained by social and cultural processes and as a social and cultural phenomenon that is constituted by a community of actively cognizing individuals. Cognitive and social processes can then be seen as complementary in that each serves as the background against which the other can come to the fore. Taking the perspective that mathematical development is a process of change in mental constructs, then cognitive analyses take as implicit the social process of acculturation. Conversely, given that mathematical learning is a process of initiation into the practice of the wider community, social analyses take for granted students' adaptive, interpretive activity.

Chapter 4

DEVELOPING A BASIS FOR MATHEMATICAL COMMUNICATION WITHIN SMALL GROUPS

Erna Yackel, Paul Cobb, and Terry Wood

This chapter and the one following describe various aspects of the interaction that occurs when children work in pairs or small groups to complete instructional activities in mathematics. The use of small-group problem solving as an instructional strategy is encouraged by current calls for reform in mathematics education (National Council of Teachers of Mathematics, 1989, 1991; National Research Council, 1989). Small-group collaborative learning is, however, much more than a way to organize the social structure of the class. Informed use of pair-collaborative learning requires that we understand the learning opportunities that arise and the constitution of the social interactions that occur when children collaborate to complete instructional activities. In addition, it is also essential to investigate the nature of the relationship between these interactions and children's individual mathematical conceptions if we are to understand how children develop the ability to communicate their mathematical ideas and explain their thinking to one another. Moreover, analysis of each of these aspects of collaborative learning must necessarily consider the social norms that have been negotiated in the classroom.

In this chapter, we focus on the development of a basis for mathematical communication among pairs as the interactive constitution of taken-as-shared ways of knowing. In the next chapter, our purpose is to illustrate how children's individual mathematical conceptions influence the nature of the social interactions that occur when students work in pairs to solve instructional activities in mathematics. At the beginning of our investigation in the second-grade classroom, we were aware of the value of collaborative group work from the neo-Piagetians and from personal experience. The use of small-group problem solving has more recently been encouraged in the reform recommendations for mathematics education (National Council of Teachers of Mathematics, 1989, 1991) as stated by the National Research Council (1989).

> Educational research offers compelling evidence that students learn mathematics well only when they construct their own mathematical understanding. To understand what they learn, they must enact for themselves verbs that permeate the mathematics curriculum: "examine," "represent," "transform," "solve," "apply," "prove," "communicate." This happens most readily when students work in groups, engage in discussion, make presentations, and in other ways take charge of their own learning. (pp. 58, 59)

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Small group interaction is seen as one way to encourage the development of mathematical relationships, reasoning, and communication and to otherwise engage students in meaningful mathematical activity.

In this chapter we examine one aspect of working collaboratively in small groups, namely, the development of a basis for communication between the partners in the group as they attempt to collaborate to solve problems.

COLLABORATIVE LEARNING COORDINATING PERSPECTIVES

An implication of constructivism for mathematics instruction is that it should be problem-centered (Cobb et al., 1991a; Kamii, 1985; Thompson, 1985). We do not mean by this that typical word problems should be the focus of instruction. Rather, we mean that children should be engaged in attempting to resolve problematic situations for themselves. Problems arise for students as they attempt to achieve their goals in the classroom. The situations that children find problematic take a variety of forms and can include (a) resolving obstacles or contradictions that arise when they attempt to make sense of a situation in terms of their current concepts or procedures, (b) accounting for a surprising outcome (for example, when two alternative procedures lead to the same result), (c) verbalizing their mathematical thinking, (d) explaining or justifying a solution, (e) resolving conflicting points of view, (f) developing a framework that accommodates alternative solution methods, and (g) formulating an explanation to clarify another child's solution attempt. As these examples make clear, genuine mathematical problems can arise in the course of social interactions as well as from an individual child's attempts to complete an instructional activity.

We have already suggested that children engage in two types of problem solving as they work together in small groups. On the one hand, they attempt to solve their mathematical problems, and on the other hand, they have to solve the problem of working productively together. From our theoretical perspective, consensus seeking and genuine attempts to communicate are part of collaborating to learn. Once social problems have been temporarily resolved, the interactions that take place give rise to opportunities for learning that result directly from the interactions. These opportunities arise from the children's genuine attempts to develop a mutual basis for mathematical communication and from their interpretations of each other's mathematical activity as they work to resolve problems they encounter.

Both the constructivist and the social interactionist perspectives are useful in analyzing the process by which children develop a basis for mathematical communication. From the constructivist perspective, meaning is not inherent in written or spoken language. It is constructed by each individual on the basis of his or her own experience and involves individual interpretation. This view has been expressed by Odgen and Richards (1946) in a treatise on meaning, as follows:

> Words, as every one now knows, "mean" nothing by themselves, although the belief that they did was once equally universal. It is only when a thinker makes use of them that they stand for anything, or in one sense have "meaning." (p. 10)

The consequence of this perspective is clearly stated by von Glasersfeld (1988a) when he said, "The meanings of whatever words one chooses are one's own, and there is no way of presenting them to a reader [or listener] for inspection." (p. 26). If this is the case, how can two individuals communicate? What is the basis for their communication? Von Glasersfeld (1983) provides some guidance on this point.

> If the meanings of words are, indeed, our own subjective construction, how can we possibly communicate?...The trouble stems from the mistaken assumption that, in order to communicate, the representations associated with the words that are used must be the same for all communicators. For communication to be considered satisfactory and to lead to what we call "understanding," it is quite sufficient that the communicators' representations be compatible in the sense that they do not manifestly clash with the situational context or the speaker's expectations. (p. 53)

In this case, we say the communicators have taken-as-shared interpretations. Miscommunication occurs when there are discrepancies in the individual interpretations of would-be communicators. When the individuals become aware of such discrepancies in their interpretations, it is possible for them to address the differences and thereby develop a basis for communication.

The constructivist view of meaning is compatible with the symbolic interactionist view. Blumer (1969) describes it as follows:

> It [symbolic interactionism] does not regard meaning as emanating from the intrinsic makeup of the thing that has meaning, nor does it see meaning as arising through a coalescence of psychological elements in the person. Instead, it sees meaning as arising in the process of interaction between people. The meaning of a thing for a person grows out of the ways in which other persons act toward the person with regard to the thing. ... Thus, symbolic interactionism sees meanings as social products, as creations that are formed in and through the defining activities of people as they interact. (pp. 4, 5)

According to this perspective, we can say that because children work collaboratively to complete instructional activities, the mathematical meanings that arise between them are formed as part of their interactions with their working partners. As such, the potential for developing a basis for communication between partners is greatly enhanced.

The learning opportunities that arise from children's attempts to communicate with each other include those that arise not only as they attempt to resolve conflicts but also as they verbalize their thoughts in the course of a dialogue and as they attempt to interpret and make sense of their partner's verbalizations. These learning opportunities arise naturally in the course of dialogues characterized by a genuine commitment to communicate (Rommetveit, 1985). Collaborative discourse can help children clarify their own understandings by talking (Levina,

1981) and by reconceptualizing their own cognitive constructions as they attempt to make sense of their partner's explanations. Consequently, our initial inclusion of small-group collaboration as an instructional strategy was a deliberate attempt not only to facilitate the occurrence of situations that children find problematic but also to encourage communicative dialogue that affords learning opportunities for children. For a discussion of the teacher's role in developing collaborative dialogue see Wood and Yackel (1990).

When children work in small groups with peers on instructional activities, their speech is characterized by what Barnes (1976) calls "exploratory talk" as opposed to "final draft talk."

> In exploratory talk and writing, the learner himself takes responsibility for the adequacy of his thinking; final-draft talk and writing looks towards external criteria and distance, unknown audiences. (pp. 113–114)

Or as Cazden (1988), following Barnes, describes it, exploratory talk is "speaking without the answers fully intact" (p. 133).

Elsewhere we have elaborated three types of learning opportunities that arise when children strive to communicate as they work together to complete mathematics activities. These include (a) opportunities to verbalize their thinking, explain or justify their solutions, and ask for clarification; (b) opportunities to reconceptualize a problem and thus construct a framework for alternative solutions; and (c) opportunities to analyze erroneous solution methods and provide clarifying explanations. The last two types of opportunities are especially likely to occur in attempts to resolve conflicts.

DEVELOPMENT OF A BASIS FOR MATHEMATICAL COMMUNICATION IN SMALL GROUPS

To illustrate the manner in which a basis for mathematical communication is interactively constituted, we present the following episode that occurred early in the school year in which a pair of children, John and Andy, are working on an activity page of thinking-strategy word problems. The page has eight word problems that are sequenced so that it is possible to solve the problems by relating them to each other. The problem under discussion is, "After Mary gave away 5 stickers to Julie, she had 7 left. How many did Mary have to begin with?" The episode begins as follows:

John: 7, 8, 9, 10, 11, 12, (pointing to five different locations on the numeral 5 in the problem statement as he counts).

Andy: No.

John: See, 7, 8, 9, 10, 11, 12 (again pointing to the 5 and writing 12 on the paper).

Andy: Come on. We gotta do this together.

John: (Reads the next problem.) "Sam had 12 guppies. Five of them died. How many were left?" What did he have? What are guppies?

Andy: I don't—I don't know. We've gotta read together. 1, 2, 3, 4, 5 (pointing to the 5 in the previous problem) 6, 7, 8, 9, 10, 11, 12, 13—(pointing to the 7 in the previous problem).

In this segment of the episode, John solves the problem quickly by counting on. Our knowledge of the children's mathematical conceptions indicates that both children are capable of using counting-on solutions. Nevertheless, Andy was not a participant in this solution process and is not ready to accept John's solution without figuring it out for himself. The solution Andy uses, namely, to count up to 5 from 1 and then to count on 7 more indicates that he interprets the problem differently from John.

The method that each child is using to keep track of the number of times he is counting is to "touch point count" five places on the numeral 5 and seven places on the numeral 7. At the beginning of the school year, several children in this classroom used touch point counting, which they had been taught to use in their first-grade class. This method is subject to the type of error Andy made here. It is necessary to remember how many places you are to point to for each different numeral. Andy pointed to eight places instead of seven on the numeral 7. In fact, it is apparent from the hesitation in Andy's voice when he first says 13 that he does not know whether or not he should stop his counting activity. We interpret John's next move as a direct response to Andy's hesitation. The subject of the next exchange is to clarify the result of 7 plus 5. John sets out to do this by using a different approach to demonstrate his counting. Notice that John continued to discuss 7 plus 5 rather than 5 plus 7. We can conclude there was no need for the children to discuss this issue. The fact that the sum is the same is part of what is taken-as-shared by both children and hence did not become a topic for discussion. As John continues, he abandons the method of pointing to locations on the numerals, the method that led to Andy's erroneous solution, and uses his fingers instead. Further, he adds a step at the beginning of the explanation.

John: 7. See, 7 (shows 7 on his fingers).

Andy: 8, 9.

John: (Again shows seven fingers briefly and begins to count on his fingers) 8, 9, 10—8, 9, 10, 11, 12 (stops counting when five fingers are up).

Andy: (simultaneously and subvocally) 11, 12.

Andy: 13.

Instead of starting from 7 and counting on, this time John begins by showing seven fingers, with the explicit instruction to Andy to take note, "7. See, 7." Through this move he is attempting to help Andy develop a basis to interpret his next action, namely counting on from 7. John then removes the seven fingers and continues his counting, putting up one finger with each number word, until five fingers are up. Andy counts along subvocally up to 12 but is not looking at John's fingers. He appears to be doing some mental counting of his own. We have no way of knowing why Andy again said 13.

At this point, John puts forth a third explanation. This argument is of a different character than the previous arguments. In the first two instances, John is explaining how to find the sum of 7 and 5. Now John takes Andy's answer of 13 and gives an explanation of why it leads to a contradiction.

John: 13 is 6 (puts up a finger on his second hand as he says this).

Andy: I don't ... (pauses and subvocally says 4, 5). Yeah.

From the observer's perspective, the argument John is making is that in order to get 13 it is necessary to add 6 to 7, and because the problem was to add 5 to 7, it is not possible to get 13. Notice the brevity of John's explanation, "13 is 6." This statement, taken alone, is nonsensical. However, in the context of the ongoing discussion, it is completely sensible. John assumed that Andy would interpret it in a manner compatible with his own intentions. This is consistent with the way individuals typically develop interpersonal communication. Each party assumes that the other will interpret his or her communication attempts, verbal and nonverbal, in a manner consistent with his or her intentions. This is what we mean by a taken-as-shared basis for communication. It is only when there is an apparent lack of fit between the interpretations that it is necessary to provide more clarification. In this instance we have some, albeit limited, evidence that Andy's interpretation of "13 is 6" was compatible with John's intent. Andy subvocally utters 4, then 5, then says, "Yeah," and writes down 12. The episode continues with the next problem: "Sam had 12 guppies. Five of them died. How many were left?

John: Sam had 12—

Andy: Sam had 12 guppies. 5 of them died....

John: What are guppies?

Andy: ... died. How many were left? Had 12. 5 died.

John: 12 take away 5.

Andy: (subvocally) 11, 10, 9, 8.

John: Okay.

Andy: 7, 8.

John: 7. Yeah, it's 7. 7. (He uses his fingers to figure this out, but it is not possible to infer his thinking. Meanwhile, Andy writes 7 on the paper.)

The apparent lack of communication here is, in fact, an illustration of the smooth flow of collaborative activity when there are no apparent discrepancies in the children's interpretations. The next problem is the following: "Daisy Duck invited 14 boys and girls to her birthday party. If 7 of them were boys, how many were girls?"

John: She invited 14, and 7 of them were boys. So 7 girls.

Andy: So 7 of them were girls. (Andy writes 7 on the paper.)

It appears that the children have completed this problem, so John's next comment comes as a surprise. We can only speculate that he stopped to reflect on the solution. His further comments indicate that he is unsure of the sum of 7 and 7.

John: So what's it gonna be?

Andy: It's 12, 12, 7, 7 (commenting on the four answers on the page).

John: 7 plus 7.

Andy: (Turns paper over). Now we have to do this side.

John: Wait (turns paper back to the first side). Wait, wait, wait. Okay, what is it?

At this point John is questioning the solution he proposed and Andy agreed with. Apparently he is not sure that 7 plus 7 is 14 and wants to figure out what the sum is. But Andy is convinced that 7 is the answer.

Andy: 7.

John: 7 plus 7. (Andy turns the paper back over and John again turns it back to the first side.) Wait, wait, wait. Okay. See, 7 plus 7 equals—I don't know.

Andy: 14.

John: 13.

Andy: 14.

John: 7 plus 7 equals 13.

Andy: 14.

John: So there must be 8 boys.

A conflict has arisen in that the children have different answers for the sum of 7 and 7. The strategy each child is using, of simply repeating his own answer, produces a stalemate. There is no basis from which to proceed to resolve the disagreement. John seeks to resolve the stalemate by initiating a statement that goes beyond a restatement of his answer to the sum of 7 and 7, but it does not address the conflict. He attempts to produce an argument that uses his result to draw a conclusion: "So there must be 8 boys." John's argument is this. If 7 and 7 is 13 then 7 and 8 is 14, so in order to have 14 children at the party the number of boys would have to have been 8, but it was given as 7.

Because John's statement does not address the original conflict, Andy ignores it. Each child goes back to simply reiterating his own answer. Notice that while Andy continues to repeat 14, his answer to 7 plus 7, John now repeats the "apparent" contradiction.

Andy: 14.

John: 8 boys. 8 boys.

Andy: No.

John: Yes. It's 8 boys (erases the 7 written on the page.) Okay, see.

Here again we see John initiating an attempt to develop a basis for communication with Andy. The, "Okay, see" indicates that he is about to explain his thinking and is inviting Andy to consider his perspective. His attempt is interrupted.

Andy: You don't know...

John: 7 plus 7 is...

Andy: 14.

John: 13.

Andy: 14.

John: 13.

We can only speculate that John was going to give an explanation that made use of the "fact" that 7 plus 7 is 13, and so his attempt was thwarted by Andy's refusal to accept that 7 plus 7 equals 13.

Andy: No, it isn't. It equals 14.
John: Okay, see 10, 11....

At this point Andy takes the initiative in attempting to develop a basis for communicating about their individual mathematical activity. He suggests using their fingers to count 7 and 7.

Andy: Like 7 (shows seven fingers by using all the fingers of his left hand and the thumb and first finger of his right hand). Then you put your 7 more.

This suggestion is interesting in light of their activity while solving the first problem in this episode. In that instance Andy's first solution attempt involved counting all. His proposal here is to do the same. This suggestion is acceptable to John, who shortly puts up seven fingers himself. We note here again that both of these children are capable of counting on. In fact, they used counting-on solutions for problems they had just solved. However, here this was a genuine dispute, and they went to a more primitive method, namely, counting all. This reduction to a primitive solution method that is undisputed by all participants is something we commonly observed in whole-class discussions as well as in small-group discussions. In most cases, as here, there was no explicit discussion of what would be a mutually acceptable basis for agreement. This basis is taken for granted by both partners. It is a taken-as-shared basis for communication. Unless and until there is reason to believe that each participant has a different interpretation of the situation, the partners can effectively communicate. That is, they each assume that their interpretation of the situation is shared by the other. Their interpretations are taken-as-shared. As the dialogue continues, it becomes apparent that here the participants have differing interpretations of each other's finger patterns for 7 (see Figure 4.1). From the observer's perspective, their interpretations are parallel, not equivalent. Andy's finger pattern for 7 is one full hand and the thumb and first finger on the other hand. John's finger pattern for 7 is one full hand and the first and second finger on the other hand, with the thumb folded down. John, on seeing Andy's finger pattern, interprets it in light of his own use of finger patterns. He ignores Andy's thumb and interprets Andy's fingers as showing a pattern for 6.

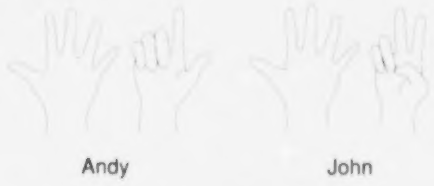

Figure 4.1. Finger patterns for seven.

John: It's 6 plus 6. No, that's 6. (Points to Andy's finger pattern. At this point John raises the middle finger on Andy's right hand.) That's 7.

Andy: Okay. You put 7 down. (John puts out all the fingers of his left hand and the first and second fingers on his right hand.

John: 1, 2, 3, 4, 5, 6, 7 (counting his own fingers) 8, 9,... (counting two of the fingers on Andy's right hand)

Andy: Hey, wait. You're putting out 2 fingers.

Andy's interruption here comes as he reflects on John's counting. John counted two fingers in addition to a full hand of his own and now has counted two of Andy's fingers but has one more to count before going to the full hand. Andy has noticed the discrepancy between the number of fingers he has displayed and the number John has displayed.

Andy: You need 3, too. Hey, not all of them (as John spreads out his whole right hand).

John: (Disagreeing) Huh-uh.

Andy: You need 3.

John: That's 8 fingers. That makes 18. (Andy counts his own fingers and goes back to his original finger pattern. John again shows his finger pattern for 7 and counts all).

John: 1, 2, 3, 4, 5, 6, 7, ... 12, 13, 14 (counting all fingers starting with his own and then Andy's).

Andy: So it's 7. (Andy writes 7 on the paper.) That's 7.

John: How do you know? Take away....

Andy: Just like....

John: Yeah.

Andy: 7 plus 7 is 14.

John: 7 take away 7 is....

Andy: 7, so ... (turns the paper over and begins to read the next problem).

We interpret John's last comment as an error in speech. He apparently means 14 take away 7. Andy interprets the remark this way and responds to what we infer John intended to say. In this brief exchange John has come back to his former argument, in which he attempted to communicate a relationship between finding the sum and finding a difference, when he said, "13. So there must be 8 boys." Andy's reply, "7, so...," indicates that John may have been successful in this communication attempt.

The collaboration in this episode is especially productive in that, throughout, each child is attempting to make sense of the other's mathematical interpretations and activity. To do so requires that they distance themselves from the activity in order to extend their own interpretation and thus make sense of the other's activity. The subject of the conversation at times revolves about differences in interpretation, as in the case of the discussion about the finger patterns. John interpreted Andy's finger pattern for 7 as a finger pattern for 6. In the final resolution of the conflict each child used his own (preferred) finger pattern without dispute. A basis for mathematical communication had been established, and the children's interpretations of their actions (including their finger patterns) were no longer parallel. They were now equivalent.

The next brief episode shows the same pair of boys working together 4 months later in the school year. This episode illustrates that the collaborative activity is productive and proceeds smoothly. We contend that in this instance the children's individual interpretations are equivalent. Consequently, the amount of dialogue is relatively limited. There is little need to explain or justify because each child takes it for granted that the other has interpreted the mathematical situation in the same way. The children are working on multiplication problems in the balance format. The problems are shown in Figure 4.2.

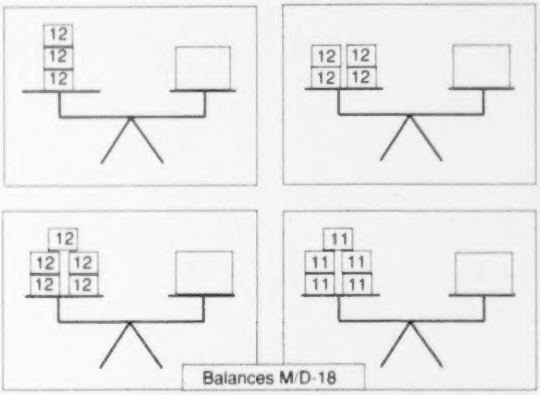

Figure 4.2. Multiplication problems in the balance format.

In the episode given here the two children are discussing the solution to the first problem, 12, 12, 12: ___.

Andy has written in 36 as the answer without comment.

John: 32. Andy, that's 32 ... 32.

Andy: What?

John: 32, 'cause 10 and 10 make 20 plus, and those 2's right there would make....

Andy: (Interrupts) 10, 10, 10, 10, 10, and 10 make 20, right. Two more make 24.

John: (Interrupts) 2, 3, 4, 5, 6 (counting the 2 in the 12's).

Andy: Just 10...

John: 12 more than...

In the conversation between these children they both talk about 10s and 2s, without explicitly discussing that 12 can be thought of as 10 and 2. The concept of 10 as a unit that is simultaneously one 10 and ten 1's is well established in this group. Each child is able to think of numbers such as 12 as being made up of 10s and 1's and so can operate on the 10s and the 1's separately. This does not need to be an explicit topic of their conversation. When one child uses language that indicates that he is thinking of the numbers in that manner, the other child is able to make sense of it as well. Their taken-as-shared interpretations are equivalent. Consequently the amount of conversation as the problem is solved is relatively limited.

The episode continues as Andy explains to John how he thought about each of the last three problems, beginning with 12, 12, 12, 12: ___.

Andy: 12 and 12—that's—makes 40, right?

John: I know. I know.

Here John's "I know" indicates that his interpretation of the task is compatible with Andy's, which is to think of the 12s in terms of 10s and 1's.

Andy: 42, 44, 45, 48 (pointing to the third task). That makes 52, 54, 56, 58, 60 (pointing to the fourth task). And 11 and 11 make 22, and 11 and 11 and 22 plus makes...

John: 44.

Andy: 44 and 11 more make 55.

John: Yeah. I'll go and get the next one [activity page].

The dialogue confirms that the children have achieved a genuine basis for mathematical communication. Their equivalent taken-as-shared interpretations include that numbers can be partitioned into 10s and 1's when desired and can also be thought of as composed of single units, as when Andy says, "11 and 11 make 22 ... 11 more make 55." Both children's mathematical concept of number makes this type of flexibility possible without explicit reference to it.

The power of these taken-as-shared interpretations for collaboration on the activities becomes apparent by considering the activity of another pair of children working on the same balance format activity page for whom partitioning numbers into tens and ones was not a taken-as-shared interpretation. One of the partners in this group solved the balance 12, 12, 12, 12: ___ by putting out four groups of 12 multilinks, with each 12 shown as one full bar of 10 and two 1's. She counted up the total as follows, "10, 20, 30, 40, 41, 42, 43, 44, 45, 46, 47, 48." Her partner rejected her method and insisted on counting up by 1's all the way from 12 to 48. For him, each 12 had to be considered as a whole unit and in sequence. His interpretation of the problem, based on his own mathematical construction of numbers, did not include the possibility of finding the sum by partitioning each 12 into 10 and 2 and adding up the four 10's and then the four 2's.

It might seem that the notion of a taken-as-shared basis for mathematical communication implies that when children work together to complete instructional activities, the level of their activity must necessarily be determined by the child who is less conceptually advanced. Our experience is that this is not necessarily true. When it becomes apparent to one of the partners that there are discrepancies in their individual interpretations, one possible solution is to revert to a more basic level. However, another alternative is for the child who is aware of the discrepancy to make it an explicit topic of conversation. In the latter situation, both partners have a unique opportunity for learning. On the one hand, the partner who is aware of the discrepancy has an opportunity to extend his or her own conceptual framework to try to make sense of the partner's interpretation and to formulate an argument or explanation that might be meaningful to the partner. On the other hand, the other partner has the opportunity to be made aware of the discrepancy and the possibility for an alternative interpretation. A detailed example of this is given in Yackel et al. (1991).

SUMMARY AND DISCUSSION

We have argued that when children work collaboratively in small groups to complete instructional activities in mathematics, they must have a basis for mathematical communication. The process of the development of such a basis can be observed when discrepancies in their individual interpretations become apparent. What has previously been assumed by the participants to be taken-as-shared then becomes an explicit topic of discussion. Differences are then either clarified or the discussion moves to more basic levels of understanding, until a point is reached where the participants' interpretations are compatible. Thus, children's individual mathematical constructions influence their interpretations and hence are intimately related to the basis for communication that is developed between children. When a basis for communication is achieved, then the collaborative activity proceeds smoothly and the dialogue is relatively limited. Learning opportunities occur for children both as they attempt to resolve conflicts and as they build on each other's activity as they meaningfully interpret each other's actions and comments (Yackel, Cobb, & Wood, 1990; Yackel et al., 1991).

Chapter 5

THE RELATIONSHIP OF INDIVIDUAL CHILDREN'S MATHEMATICAL CONCEPTUAL DEVELOPMENT TO SMALL-GROUP INTERACTIONS

Erna Yackel, Paul Cobb, and Terry Wood

This chapter elaborates our understanding of small-group problem solving in elementary school mathematics by focusing on the relationship between children's individual mathematical conceptions and the nature of their social interactions as they work together in small-group problem-solving settings. In particular, the manner in which children's individual mathematical conceptions influence the nature of the social interactions that take place is explained. Previously we have focused on the importance of the social interactions that occur in small-group settings in proving unique opportunities for learning (Yackel et al., 1991) and on the negotiation of classroom social norms that facilitate collaboration. As we analyzed small-group interactions, we were able to trace the development of cooperative activity between partners. Initially, children typically divided the labor, taking turns as they completed the activities. Under the guidance of the teacher, social norms for small-group activity were negotiated that resulted in children reconceptualizing their understanding of work cooperatively from division of labor to working together to complete the instructional activities. Working together came to be understood as explaining your thinking to your partner, listening to your partner's explanations, and jointly developing a solution. Thus, situations in which children failed to work collaboratively were interpreted as situations where the participants were violating the social norms. However, as we continued to analyze the small-group data, we found that some social interactions that we had previously explained only as violations of social norms could be further interpreted by taking the children's individual mathematical conceptions into account. The analysis shows that children's conceptions influence not only the nature of their individual and collaborative mathematical activity but also the nature of the social interaction itself. It is this relationship between individual children's mathematical conceptions and the nature of the social interaction that is elaborated in this chapter.

Steffe and colleagues (1988; 1983) developed models of children's arithmetic concepts by analyzing individual interviews and one-on-one teaching sessions in which children were presented with a variety of tasks they might solve using

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various methods including counting. The models that resulted from this research are useful in that they allow one to infer how a child might be interpreting and attempting to solve an instructional activity in the classroom on the basis of his or her actions during a prior clinical interview. This in turn helps to provide an understanding of the degree of sophistication of a child's arithmetical concepts. For example, according to Steffe's conceptual models, a child who can solve a subtraction problem such as 23 take away 18 by "counting down to," that is, counting down to 18 from 23 and answering 5, makes a more sophisticated interpretation of the task than a child who can only solve subtraction problems by "counting down from," that is, by counting down 18 times and ending at 5.

Similarly, Steffe et al. (1988) and Cobb and Wheatley (1988) have extended the counting types model to describe qualitative differences in children's concepts of ten. Cobb and Wheatley found that children's interpretations of a task were highly contextual. Accordingly, we do not label children as being at a specific level of development with regard to concept of ten, but instead focus on children's ways of operating with ten in specific situations. In comparison with other concepts of ten, a concept of ten as an iterable unit that is simultaneously one 10 and ten 1's is relatively sophisticated (Cobb & Wheatley, 1988; Kamii, 1986; Steffe et al., 1988). The analyses of episodes presented in this chapter were informed by the work of Steffe and Cobb and Wheatley.

THE NATURE OF SMALL-GROUP ACTIVITY

Our approach to small-group collaborative work differs from other approaches in which children have differentiated roles or in which the group as a whole achieves a goal by having each group member complete one or more subordinate tasks (Cohen, 1986; Dees, 1990; Johnson & Johnson, 1975; Slavin, 1983). Instead, our approach to small-group work attempted to make mathematical activity itself the focus of the children's cooperation. In this approach there is not an extrinsic reward system. The nature of the tasks and the classroom social norms make extrinsic rewards unnecessary. In such a setting, persisting on a challenging task is valued over getting correct answers or completing a large number of tasks. The success of the approach is a result of the social norms that are negotiated in the classroom (Cobb et al., 1989; Cobb et al., 1988; Yackel, Cobb, & Wood, 1989; Yackel et al., 1991).

The norms for cooperative group work that were negotiated by the children and the teacher included the obligations to work together to solve the problem, to reach a consensus (i.e., to agree on an answer to the problem), to explain their thinking to their partner, and to listen to and try to make sense of their partner's explanation. The latter might include challenging their partner's explanations, asking for clarification, explaining why they disagree with their partner's solution, and taking the perspective of their partner. The process of explaining one's thinking to one's partner can involve organizing or reorganizing and reformulating one's solution process in order to verbalize it, reconceptualizing one's solution in order to provide alternative explanations, and distancing oneself from one's thinking to try to take the perspective of another. The process of engaging in collaborative dialogue involves developing mathematical communication on the basis of commonly shared mathematical activity.

Norms for children's individual activity had also been mutually negotiated. These included that children were expected to attempt to solve the problems in ways that were personally meaningful to them and that they were responsible for explaining their solution attempts. These norms were important not only for each individual child's learning but also for the class discussion, in which both partners were responsible for providing an explanation.

Clearly, there are possible tensions between the norms for individual activity and for small-group activity. For example, if one partner wants to explain his or her solution method while the other child wants to think through his or her own method, then one of the children must temporarily suspend the attempt to fulfill his or her obligation. For example, the child who wants to explain may have to wait until the partner has had an opportunity to think through his or her own method. Typically, this is a situation of give and take, with children offering explanations while their partners are thinking about their own methods. Children then have the option of ignoring their partner's explanation and focusing on their own solution process or of attending to the partner's explanation and using it to develop a joint solution. Such collaborative activity has the potential to facilitate children's learning in ways that are not possible in traditional instruction. For a discussion of the unique learning opportunities that arise in the course of collaboration in small-group problem solving, see Yackel et al. (1991).

It is equally important to develop an understanding of those situations in which small-group activity did not appear to be collaborative because children chose to work independently, perhaps because one of the children was not actively engaged in solving the task or because the children were unable to resolve a conflict. As we analyzed a sampling of videotapes of two pairs of children who maintained their partnership for a majority of the school year, we observed that we could frequently explain such puzzling situations by taking the children's current mathematical conceptions and interpretations of the activities into account.

During the course of the teaching experiment the classroom videotapes were reviewed on a daily basis for the purpose of informing the continuing development of instructional activities. At the same time we were able to obtain information about the children's mathematical activity and the social interactions of the children with each other and with the teacher. Subsequently, selected videotapes of small-group work for two pairs of children have been analyzed to further define and clarify appropriate methods for in-depth microanalysis. The observations made in this paper are the result of the intermediate stage of analysis. In this stage we sought to relate the children's interactions to the solution methods they were using and to their individual mathematical histories. This approach proved useful in that it allowed us to explain some of the social interactions that previously were puzzling. In particular, for one pair, John and Andy, the nature of the interaction changed significantly when one of the partners did not have a way to complete a problem. In another pair, Ann and Ron, one child often used a standard algorithmic method for adding two-digit numbers that she was unable to explain to her partner in a way he could understand. As a consequence, on a number of such occasions her partner actively resisted arriving at a consensus. In the remainder of this chapter, we discuss the nature of the relationship of these two pairs and present illustrative examples from their small-group activity.

EXAMPLES FROM THE CLASSROOM

Ann and Ron

The first example we present is of two children, Ann and Ron. In the beginning of the school year, information from their clinical interviews revealed that both Ann and Ron used counting-on methods to solve addition problems. However, Ann had developed an elaborate finger pattern scheme that permitted her to quickly and accurately solve problems involving two-digit numbers. She used these patterns to solve tasks that were designed to gain information about her understanding of the concept of ten. In contrast, Ron counted by ones to solve these same tasks. The importance of this for us is that from the start, Ann had efficient computational procedures that were not easily comprehensible to an observer or a partner.

In the second interview, which took place in January, differences in Ann's and Ron's understanding of ten were apparent. Both children determined the answer to the question, "How many times do you have to count to get from 33 to 43?", by counting. Ann got 10 for an answer, and Ron gave 13 as his answer. When asked how many rows of ten can be made from 43 blocks, Ann had no idea of how to solve the problem. Ron, after giving an initially confusing explanation, later came back to the task and immediately answered, "Four." Despite Ann's limited concept of ten she could readily arrive at answers to two-digit addition problems. She used the standard algorithm, which, she told us, had been taught to her by an older friend. Ron solved 16 + 9 by counting on by ones but was unable to solve any of the addition problems involving 2 two-digit numbers. His method of counting by ones did not include a way of keeping track of how many times he had counted when the numbers involved were large. In this class, Ann was one of two children who used the standard algorithm, whereas Ron was one of seven children who had not yet constructed a way of solving two-digit problems. In this second interview, as in the first, we have evidence that although Ann's concept of number and of ten was no more advanced or sophisticated than Ron's, she once again had a more efficient scheme for arriving at answers than Ron. We will later argue that Ann's understanding of the standard addition algorithm was only procedural.

Instructional activities used in the months of January and February focused on development of the concept of ten. These were designed to encourage children to elaborate their concept of ten and their methods of operating with two-digit numbers. The differing interpretations Ann and Ron gave to these activities significantly influenced their ability to collaborate when they worked together. Ann was able to use the standard algorithm successfully and arrive at correct answers, but her explanations in these cases were limited to repeating the steps of the procedure. Her explanations did not make sense to Ron, who was not yet able to conceptualize ten as an abstract unit simultaneously composed of one 10 and ten 1's. Ron's concept of ten was just emerging, and he needed figural material with which to operate. The method he typically attempted to use was to draw pictures of ten-bars and of single units and use these as material for counting and calculating. On the one hand, Ron was unable to relate Ann's activity to his own interpretation of the tasks. On the other hand, Ann knew that she was generally able to produce correct answers using her method and so had little incentive to attend to Ron's solution attempts. We note, parenthetically, that the difficulties caused by Ann's use of the algorithm have since been resolved in project classrooms by requiring that when children use the standard algorithm they give mathematical justifications for the algorithmic procedure.

The following episode, taken from a class session in the middle of February, illustrates the nature of the dilemma that occurred between Ron and Ann when she used the standard addition algorithm. In this episode, the children had been working on multiplication problems in the balance format (Figure 5.1) and had completed 12, 12, 12, 12:___ using independent methods. Ann arrived at an answer of 48. Ron's activity suggests that he also arrived at an answer of 48. However, he insisted that 47 was the answer to the problem. The episode begins as they start to work on the second problem: 12, 12, 12, 12, 12: _____.

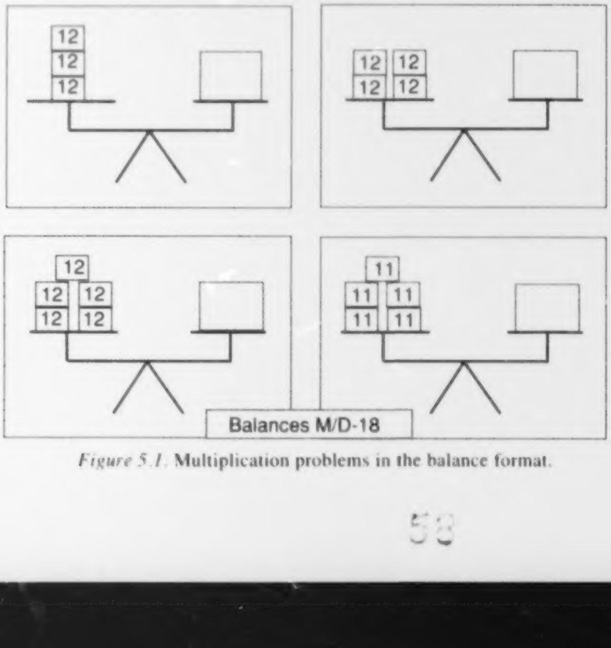


Figure 5.1. Multiplication problems in the balance format.

Ann: On this one you better not disagree. (Ann adds 5 twelves vertically using the standard algorithm.) I believe it's 60. And you'll say it's 59.

Ron: Will I?

The children briefly discuss their previous disagreements and their abilities to solve the problems. They then return to solving the problem.

Ann: Now listen.... 2, 4, 6, 8, 10. You put the "o" there [under the column of 2s] and the 1 there [above the column of 1's]. 1, 2, 3, 4, 5, 6. Do you agree that's 60?

Ron: (Long pause) ... I didn't agree yet. I'm still counting to see if it's right.

Ron's comment that he doesn't agree yet is consistent with his understanding that in this class you are obliged to figure out problems in ways that are personally meaningful. Although Ann's method was meaningful to her, it was not to Ron. In his next remark we see, however, that he is attempting to make sense of Ann's method. He begins in the same way she did by adding up the five 2s to get 10.

Ron: Hold on. This is 10. Look. 1, 2, 3, ..., 8, 9, 10 (adding up the five 2s.)

Ann: One there [above the column of 1's]. 1, 2, 3, 4, 5, 6, 60.

Ron: 610?

Ann: You don't, you don't listen.

According to Ann's perspective, Ron has just participated in an explanation of her solution method. He added up the five 2s in the 1's column, and she continued on with adding up the 10s column. For Ann, repeating the steps of a procedure constituted an explanation. Her "You don't listen" indicates her assumption that Ron's failure to understand is merely a problem of not paying attention. Ron's next remark indicates his attempt to communicate to Ann that he does not understand mathematically how her procedure works. He makes it clear to her with his "um-hum" and "yeah" that he is listening to her explanation.

Ron: I don't get this.

Ann: Look....12 plus 12 plus 12 plus 12. How many 12s are there? (She writes the numbers out in vertical form). 5? (Subvocally counts 5).

Ron: Um-hum. See 1, 2, 3, 4, 5 (counting the number of 12s in the problem).

Ann: 2, 4, 5, 6, 10. Now listen. Put that "o" there....

Ron: Yeah.

Ann: The 1 up there.

Ron: That isn't "o."

Ann: 1, 2, 3, 4, 5, 6.

Ron: That's not "o." That's 10.

Ann: No.

Ron: That isn't "o," though. That's 10. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

Ann: Now listen.

At this point Ron goes off to get the teacher. Ron recognizes that he and Ann have arrived at an impasse and outside intervention is needed.

Ann: (To the teacher) He doesn't—he doesn't get—he doesn't get that right.

Ron: (Simultaneously) I don't understand. All the 2s add up to 10. She takes 1 away from the 10.

Teacher: Because (pause) it's a long story.

Ann: 2 plus 2. 2, 4, 6, 8, 10. I know that's 10.

Ron: It is 10.

Ann: You put the darn "o" there. 1, 2, 3, 4, 5, 6.

Ron: (Simultaneously) Show me how you're doing it. I don't understand.

The conflict here between Ann and Ron is unresolvable. Both children interpret the activity as one of adding five 12s. However, Ann's interpretation of how to do so is to carry out the steps of her algorithmic procedure. Her repeated use of "o" in talking about the digit zero in the number 10, in spite of Ron's repeated protest that "that's not 'o,'" provides convincing evidence that she is using a rote procedure rather than operating meaningfully with numbers. Ann, for her part, is attempting to fulfill her obligation of explaining her solution, as we can see from the number of times she reports the description of her procedure. She attributes Ron's failure to understand to his not listening or not paying attention. Ann's belief about the nature of mathematical activity and what counts as an explanation are confounded with each of the children's differing mathematical conceptions. Ron's resistance to agreeing with Ann might be interpreted as blatant stubbornness. We contend, however, that it might be more appropriately interpreted as his attempt to develop a collaborative relationship with Ann based on meaningful mathematical activity. From his perspective, Ann was failing to meet her obligation of justifying her solution.

The impasse the children have arrived at in this episode can be explained by considering each of the children's individual mathematical conceptions. On the one hand, Ron's inability to deal with ten as an abstract unit that is simultaneously one 10 and ten 1's makes it impossible for him to make sense of Ann's use of the standard algorithm, even if she had given an appropriate mathematical explanation. On the other hand, Ann's activity is entirely procedural and does not involve the construction of ten as a unit composed of ten 1's. Her inability, as revealed in the second clinical interview, to figure out how many rows of 10 can be made from 43 blocks, indicates that she would be unable, given her current understanding of ten, to explain the standard algorithm mathematically. Thus, the apparent lack of cooperation between Ann and Ron while attempting to solve this task was a direct result of the children's individual understandings of ten coupled with Ann's use of the standard algorithm. In this episode, it was impossible for the children to constitute a basis for mathematical communication, because Ann's recitation of her algorithmic procedure was all she could provide as an explanation, and Ron had no way to meaningfully interpret it. In a classroom where getting correct answers efficiently is the goal, Ann's activity would have been valued. In this classroom, where mathematical explanation and justification and making sense of each other's mathematical activity is expected, Ann's activity contributed to the pairs' failure to develop a collaborative solution.

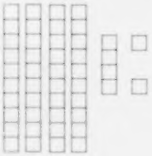
John and Andy

We further illustrate the way in which children's mathematical conceptual development influences social interaction by discussing a pair of boys, John and Andy, who also were partners for over half of the school year and generally engaged in productive collaborative activity. Even so, there were times when one of the partners was apparently not engaged in meaningful mathematical activity either by choice or because his partner controlled the use of the papers on which the activities were written. It is particularly interesting that in this case the less involved partner was not less advanced mathematically, although he was generally slower at completing calculations.

We repeatedly observed that when Andy's mathematical interpretation of an activity led to an intrapersonal conflict, given his current mathematical conceptions, then he would turn to John for assistance and the cooperative activity would resume. In many, but not all, of these cases the activity page would move to the center of the desks and the children's body positions would change so that they were turned towards each other. We interpret these movements as implicit invitations for cooperation. The importance of attending to nonverbal aspects when analyzing small-group interactions was first pointed out to us by Heinrich Bauersfeld (personal communication, February 1988).

The example that follows is taken from a class session at the end of January. The two boys are working on an activity page designed to facilitate children's elaboration of their concept of ten. In the initial clinical interviews, which were conducted in September, both boys were able to answer the question "How many times do you have to count to get from 33 to 43?" and also "How many rows of 10 can be made from 43 blocks?" By January they each had relatively sophisticated concepts of ten. For example, both boys could figure out answers to missing-addend problems involving two-digit numbers presented with hidden and visible figural material, even when the sums went over a decade. Differences in the way they typically thought about two-digit numbers are indicated by their individual methods for solving addition problems. John's method for adding 39 + 53 was, "You have 53, 10 more is 63, plus 10 more is 73, plus 10 more is 83, plus 9 ... 92." This method indicated thinking of tens from a counting-based meaning in which you add 53 and 39 by adding on by 10s and then 1's. Andy's method was, "30 plus 50 is 80 and 9 plus 3 is 12. Put all those together and I came up with 92." Andy's methods typically involved thinking of numbers as collections of 10s and 1's. Under this perspective, 39 is composed of one group of 30 and another group of 9.

With this background we present an episode in which John and Andy are working on the following activity:

How many do you have to add to  to make 73?

The activity sheet is between the boys. A researcher (Rsch) intervenes for the purpose of asking questions to document the children's solution methods.

John: (Reading the problem) How many do you have to add to....

Andy: 24! (Pointing at the task) I mean—yes 24, 24! (insisting to John) No, wait.

Rsch: (To Andy). How did you get 24?

Andy: I did it when she [the teacher] had that [overhead projector] turned on. I did it when—I did it when she had that turned on.

During the introduction to the class activity for the day, the teacher had displayed one of the activity pages on the overhead projector to make sure the children understood the intent of the activities. The problem under discussion by the boys was on the page displayed.

Rsch: Oh, okay. You tell us how you did it. I'm just interested.

Andy: Okay. Excuse me (pulling the activity page away from John and placing it directly in front of himself). Well, like, uh, 30 and 40 makes 70—and these, and 4 and 2 makes 6 so that's 76—and uh—20—20, uh—20, and 4 makes—-that only makes 60.

Rsch: Do you agree with 24, John? (John shrugs his shoulders. He cannot see the activity page.)

Andy: 40, 30 more (probably covering up three 1's of the 46) 3, 43, 43, 53, 63, 73. 73—20!

Here Andy has arrived at an interpersonal conflict. He knows that 30 is too big and 20 is too small, but he is unable to figure out a way to solve his dilemma. His ability to coordinate 10s and 1's is limited to situations where figural material is present, even if it is hidden.

John: I don't know.

Andy: (To researcher) 20. (To John) 20—do you agree? (John pulls the paper from Andy to himself.) See, 40 and 30 makes 70. Take away these 3 makes 73. Take away 3, these 3 right here. What does all that make? (Moves his fingers on the paper.) Take away these 3....

A long discussion ensues in which both children make several attempts to solve the problem. The crucial point for us here is that it was not until Andy realized that his interpretation of the task led to a personal conflict that he was unable to resolve, that he explicitly invited John to enter into the problem-solving activity. He did this both verbally, by asking "Do you agree?" and nonverbally, by allowing John to have access to the activity page so he could see the problem, particularly the diagram.

This example illustrates how Andy's interpretation of the activity and his conception of ten influenced the social interaction he had with John during this problem-solving effort. John was little more than an observer in the problem-solving process (he could not even see the problem on the activity page) until Andy encountered a conflict that he could not resolve.

The preceding analysis has not related John's mathematical conceptions or his interpretation of the task to the social interaction. John and Andy had worked together for 4 months when this episode occurred. Even though it is not likely that either of these boys could describe his partner's concept of number or of ten, they had sufficient experience working together to know that they often had different ways of interpreting and solving the problems. Thus, it was reasonable for Andy to expect that John might have an alternative way to think about the problem. The subsequent discussion, which eventually led to the solution of the problem, is very lengthy and so has not been included here. But the dialogue shows that John did interpret the problem differently, and the activity that resulted was influenced by both children's engagement in the solution attempt.

SUMMARY AND DISCUSSION

The purpose of this chapter has been to demonstrate that an understanding of individual children's conceptual development is essential if we are to understand the social interactions that occur as children work in small groups to solve mathematical activities. In particular, situations in which children fail to work collaboratively cannot always be explained solely as violations of social norms. Small-group cooperative learning in mathematics is much more than a way of organizing the social structure of the class. The interactions that occur when children work together in small groups are intimately related to the children's mathematical conceptions as well as to the social norms that have been negotiated in the classroom.

Chapter 6

THE NATURE OF WHOLE-CLASS DISCUSSION

Terry Wood, Paul Cobb, and Erna Yackel

In this chapter, we illustrate the nature of the social interaction between the students and the teacher that created opportunities for learning not found in traditional elementary school mathematics classes. In this class, the teacher and the students acted in ways that reflected their intention to genuinely communicate about their mathematical thinking and reasoning. The analysis of the discourse focuses on the class discussion that usually followed the students' work in small groups. In these settings, the purpose was to discuss the various solutions to the problems the children had just completed in collaborative settings. Class discussion also occurred at the beginning of the lesson in which the purpose was to engage students in either mental arithmetic, estimation, or spatial imaging. During these discussions, students had a few minutes to solve the problem individually and then gave their solutions to the rest of the class.

The discussions following small-group work were a frequent and regular event in the classroom and offered an opportunity to investigate the reflexive nature of teaching and learning that created opportunities for learning. However, orchestrating these discussions was a complex process for the teacher and often created dilemmas for her. On the one hand, she wanted to encourage children's individual mathematical constructions, but on the other hand, she wanted them to be acculturated into the mathematical practices of society. Thus, the process of teaching in this classroom required the development of a complex form of practice on the part of the teacher, and it sheds light on the opportunities for learning that existed for the teacher during the classroom teaching experiment (Wood et al., 1991). As we have commented earlier, in this project classroom mathematics was viewed as an activity in which students were involved in solving problem-centered activities. These activities were in contrast to traditional textbook tasks in that they were open-ended to encourage a variety of solutions from the students, which emphasized the development of underlying conceptual operations rather than standard algorithmic procedures. Therefore, it was necessary for children to be afforded opportunities to discuss their solution methods in a setting that encouraged them to explain and justify their reasoning and thinking. From a psychological perspective, they would have opportunities to reflect on their own solution methods as they explained their ways to others. Additionally, as they listened and tried to make sense of other solutions, the children might have an opportunity to reconceptualize their thinking.

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Further, from a sociological perspective, the class discussion created opportunities to learn as the teacher and students negotiated mathematical meanings that would enable the children to make connections from their individual mathematical constructions with the taken-as-shared meanings of the classroom. In such discussions it was possible for the students and teacher to interactively constitute a basis for mathematical communication that created opportunities for the teacher to guide the development of mathematical meanings that fit with those of the wider society. It is this fit between individual constructions that makes possible the subjective experience of shared, objective mathematical reality (Peirce, 1935). For the teacher and students this involves mutually establishing patterns of interaction that are necessarily quite different from those found in the traditional elementary school mathematics classes. The purpose of this chapter is to examine in more detail the nature of the discourse that occurs during two different situations during class discussion in order to provide information about the role that social interaction and communication play in the genesis of children's mathematical experiences.

PATTERNS OF INTERACTION IN TRADITIONAL MATHEMATICS CLASSROOMS

The patterns of interaction that exist in classrooms are created as an outcome of social interaction (Bauersfeld, 1980) and are beyond the conscious awareness of both the teacher and the students, but nevertheless they are jointly constructed. These patterns of interaction function as a "reliable hidden grammar" (Bauersfeld, 1988a) that creates the "smooth functioning of the class" (Mehan, 1979), which acts to reduce the complexity of the classroom processes and provides stability for the mutual activity of the teacher and students. These hidden regularities act to encourage or limit opportunities for students to engage in meaningful learning in the classroom.

An Example From a Traditional Mathematics Classroom

As a point of contrast to discussion in the project class, an example from a traditional elementary school mathematics class will be considered initially.[1] In this episode, the teacher has just finished putting 45 tally marks on the chalkboard, from which she poses an open-ended question: "Without counting, just without counting, how many sticks do you think there are there? Just (snaps her fingers), just estimate. How many do you think are there?" She then accepts students answers in a nonevaluative way.

In general, initially this lesson seems to follow an inquiry pattern in which the teacher provides an open task and then holds back direction, allowing students to offer their meanings and interpretations. The lesson appears to be about estimation, in which the students are to approximate a number for the quantity of tally marks shown. However, the episode changes abruptly with the following exchange:

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[1] From the unpublished dissertation of Betsy McNeal, 1991.

Teacher: Okay. Now, those are the estimates. Now let me show you a real easy way to figure out just how many are on the board there. Okay, let's see if we can group these together in 10s. (She counts 10 marks and circles them, continuing until 4 groups are circled with 5 left). How many groups of 10s do we have there, boys and girls?

Children: 4, 44.

Teacher: We have 4 groups of 10, and how many left over?

Children: 5.

Teacher: 5. Okay. Now how many groups of 10 did we have?

John: 4.

Teacher: We had four 10s and how many left over?

Beth: Four 10s.

Sarah: 5.

The teacher's abrupt change of the topic of the lesson from estimation serves to shift the discussion from students offering approximations to one of demonstrating the official way to get the exact number: "Those are the estimates. Now let me show you a real easy way to figure out just how many...." In making this statement, she deliberately directs the lesson to her intended purpose, which is learning about 10s and 1's. By doing so, she conveys a stance that implies a traditional form of teaching rather than one of inquiry. That is, from this view, children are assumed to have no ways of their own for solving problems (thus their estimates were ignored) and are dependent on the teacher to show them the correct way. The students, for their part in this setting, may still be trying to make sense of the teacher's actions and to figure out whether the lesson is about estimating or a specific way to count the marks accurately. As the discourse continues, the traditional classroom interaction pattern of teacher initiation–student response–teacher evaluation (Mehan, 1979; Sinclair & Coulthard, 1975) emerges, and the nature of the students' mathematical activity is reduced to supplying either the number 4 or 5 in response to teacher's questions. The teacher, reassured by the children's answers, assumes that there is mutual understanding that the purpose of the lesson is about 10s and 1's. Satisfied that the students understand, she continues:

Teacher: 5. Now can anybody tell me what number that could be? We have four 10s and five 1's. What is that number? Ann?

Ann: (Remains silent.)

Teacher: If we have four 10s and five 1's, what is that number?

Ann: 9.

Teacher: Look at how many we have there (points to the 4 groups of 10) and five 1's. If we have four 10s and five 1's, we have...? 45.

Children: 45.

Teacher: Very good.

As shown in the example, the manner in which the teacher controls and directs the nature of the interaction does not encourage discourse in which genuine communication can take place nor provide opportunities for the negotiation of meaning, and thus it does not create opportunities for students to engage in mathematical activity. Instead, the exchange becomes a ritualized interaction in which the students' opportunities to participate in the discussion are reduced to single-word responses. In this traditional form of interaction, the teacher's purpose in asking questions is to elicit the intended statement and to quickly evaluate whether the students are understanding what she had intended (Weber, 1986). Because she is only listening for responses that are correct, she continues to further assume that mutual understanding exists. Ann, however, has a different interpretation of the task that makes sense from her point of view, but for the teacher suggests that, despite her carefully sequenced instruction, Ann still does not understand. She then proceeds to reduce the ambiguity and repair the situation by providing the class with both the explanation and the justification for the correct answer. In doing so, she removes from the students any possibility of engaging in mathematical activity.

The classroom interaction described here is well known and occurs with regularity; it illustrates communication between teacher and students that is clearly asymmetrical. Voigt (1985) refers to this pattern of interaction as the elicitation pattern, which begins with dialogue in which children's opinions are elicited. However, as the lesson develops, the students' ideas are ignored in favor of the method advocated by the teacher. This initial phase is then followed by the traditional classroom elicitation-response-evaluation pattern. In this episode, the teacher's goal was not to initiate and guide the discourse so that the students could express their mathematical understanding of place value. Instead, the teacher's intention seemed to be to elicit single words from students that indicate they have understood her explanation about making groups of tens from ones and determining the resultant two-digit number. In this situation, the concept of place value has been reduced to a narrow formal statement: four tens and five ones are 45. For the children, it is not necessary to have constructed an understanding that ten can correspondingly be a unit of ten and ten ones; they need only be able to follow the teacher's verbal cues to respond with the appropriate numbers. Thus, in this classroom talking about mathematics consists of figuring out what the teacher's intentions are (i.e., estimating or exact counting by a method of grouping tens and ones) and has little to do with their own mathematical thinking. Discourse, as described here, reflects the well-known situation in traditional classes where the nature of mathematics is not open to questioning by students. Instead, the intended goal is for students to accept and understand mathematics as the teacher already knows it.

PATTERNS OF INTERACTION IN AN INQUIRY MATHEMATICS CLASSROOM

Interaction patterns that are characterized by communicative discourse and act to influence students' evolving sense of number have as a crucial feature discussions in which the children are actively involved in a process of making interpretations that involve the construction of mathematical objects. In these dialogues, students are actively involved in a form of mathematical argumentation in which explanation, justification, and elaboration are essential features.

It is understood that the manner in which the teacher acts to direct and control the dynamics of the discourse strongly influences the opportunities for students to be active participants, and it is accepted that in all classroom discourse a necessary imbalance exists between the teacher and students—the teacher modifies the discourse to a greater extent than the student (Bishop, 1985). Although it is necessary for an imbalance to exist between the teacher and the students, it is the manner in which teachers exert their influence that is important. As Guthrie (1984) states:

> Communication is an activity that requires two or more autonomous partners, one of whom may benefit from the other's skill in making the interpretation succeed, but the more one participant's output is subject to another's control, the more the discourse becomes the sole creation of the more proficient individual—and that is not communication. (p. 46)

The manner in which the teacher chooses to constrain and limit students' participation in discussion influences their opportunities and willingness to talk about their own thinking. As an example, in a study of Brazilian teachers and parents by Wertsch, Minick, and Arns (1984), differences were found in the way each group of adults interacted with children during a joint problem-solving task. The parents, for their part, tended to take control of the discourse and to provide the children with directives concerning the correct way to solve the task. As a consequence, these mothers were directly responsible for enacting the more cognitively demanding phases of the task, leaving for the children only those aspects of the problem that they could already easily carry out. The teachers, however, avoided being directive and instead encouraged the children to engage in the demands of the task by providing prompts and suggestions in their dialogue that enabled the children to solve the tasks on their own. It would appear, then, that in classroom interactions where the teacher acts to facilitate children's thinking about mathematical problems and where genuine attempts to communicate exist, students have opportunities to engage in dialogue in which they can express their mathematical thinking. In this setting, an atmosphere of mutual trust exists such that each child's opinion is respected by the others, and the teacher is necessarily sensitive to the possible potential mathematical constructions a child might make (Cobb et al., 1989; Wood et al., 1990).

It is the communicative qualities in the dialogue between teacher and students that influence the nature of the interaction that occurs and the mutual interpretation of mathematical activity. Although children construct mathematical concepts by using their own experiences and previous understandings as a guide, these concepts must eventually be compatible with the meaning constructed by others (von Glasersfeld, 1988b). When children are allowed to express their personal constructions, then the interactive constitution of mathematical meaning emerges as they verify and adjust their interpretations through a process of negotiation of meaning (Voigt, 1985, 1989). The tensions that exist between the meanings held by the teacher or by other students create the necessity to take another's perspective into account and to engage in negotiation so that meanings can be established that are then taken-as-shared. The manner in which the teacher and students mutually constitute the basis for mathematical activity in the discussion creates the possibility for establishing genuine communication discourse and creates opportunities for students to learn. It is the nature of these opportunities to learn that were created through the interplay between the social norms and the patterns of interaction and the development of communicative discourse in the classroom that will be described. The analysis of representative episodes from class discussion is guided by the theoretical perspective of symbolic interaction (Blumer, 1969), in which qualitative research and an interpretative method underlie the study (Firestone, 1987). This process differs from traditional formal discourse analysis, which was devised to reveal linguistic structures that provide form and structure to what is said. The work of Sinclair & Coulthard (1975) and Mehan (1979) fall within this tradition. Rather, the analysis presented here attempts to derive the meaning that is negotiated and thus focuses on the interpretation of the teacher's and students' meanings and actions. We are interested in analyzing the discourse for what Edwards and Mercer (1987) describe as the "understandings they convey and with how these understandings are established and built upon as the discourse proceeds" (p. 10). Thus, we are interested in the patterns of interaction, the nature of the communicative discourse, and the taken-as-shared meanings that evolve as opposed to the linguistic structures. In this analysis, attempts are made to consider both a psychological and sociological perspective of mathematical learning in which the role of action and language in individuals' learning (Steffe, 1983; Steffe et al., 1988) and the role of negotiation of meaning and argumentation in the constitution of taken-as-shared meaning (Bauersfeld, 1988a) are essential. The specific, illustrative dialogues that have been selected for microanalysis are examples of the qualities that underlie communicative discourse (Edwards & Mercer, 1987; Guthrie, 1984) and the characteristics of the interaction patterns (Voigt, 1985) in which opportunities exist for children to learn mathematics with meaning.

The two episodes that have been selected illustrate two different patterns of interaction that have been mutually constituted in the course of the class discussions. The first episode exemplifies the form the discourse takes when the discussion centers on different methods for doing the problems, and the second example illustrates the nature of discourse that focuses on the interpretation of the task. In each case, the teacher's role as she initiates and guides the discussion acts to create different opportunities for students to engage in mathematical activity.

Situations for Discussion of Solution Methods

In this episode, the class has previously been working in pairs solving two-digit addition problems using their own nonstandard algorithms; they are now reviewing their solutions in a class discussion. The problems are written on the overhead projector as follows:

$$\begin{array}{cccc} \begin{array}{r} 39 \\ +\,20 \\ \hline 59 \end{array} & \begin{array}{r} 39 \\ +\,24 \\ \hline 63 \end{array} & \begin{array}{r} 39 \\ +\,25 \\ \hline \end{array} & \begin{array}{r} 49 \\ +\,25 \\ \hline \end{array} \end{array}$$

Teacher: Now the next one, 39 plus 25. John and Dan.

John: We got 64.

Children: Agree, agree.

Teacher: Explain how you got this answer. (John and Dan have left their seats and gone to the front of the room.)

John: Well, we know that (he points to the problem) 39 plus 20 was 59.

Teacher: Okay. Do you see this over here (points to 39 + 20) 39 plus 20, they knew was 59. Then what?

John: And then 5 more would make 64.

This exchange illustrates the typical pattern of interaction that has been interactively constituted in this class, in which the teacher initially calls on a pair of students to give its solution to the problem. The pair give their answer, and the other members of the class express their agreement or disagreement. Then she asks the pair to give their method for solving the problem. At this point, if no one has questions about the solution method, the teacher calls on another pair.

Teacher: All right. Let's hear from Hanna and Andrea. How did you get this answer?

Andrea: I did it a different way than John said. You can also go like I … (she goes to the board and points to 39 + 24 and 39 + 25). Right here.

Teacher: Aha!

Andrea: These two are the same [39]. But 24 to 25 is just one more higher. Two and 2 are the same, so 64 (points to the problem).

Teacher: Great, and it saved you a lot of time, didn't it? By seeing a relationship with one of the other problems…. Let's look at 49 plus 25. Let's hear from Lesley and Peter.

Lesley: We got 74.

Teacher: 74.

Students: Disagree, agree, disagree.

Teacher: All right. Wait a minute. Let's hear what they have to say.

Lesley: 49, 49, 50, 60 (putting up 2 fingers as she counts by tens). 69, 70, 71, 72, 73, 74 (putting up fingers as she counts by ones).

Teacher: Okay. Fine. Did somebody do it a different way? Joshua and Ron? (Both boys go to the front.)

Joshua: Well, right here, 39 plus 25 was 64, and 49 instead of 39 plus 25 is 74. It's just adding 10. This is 4 instead of a 3 right here (pointing to 40 and 30 respectively).

Teacher: Aha. Mark noticed the same thing, didn't you, Mark? (Mark nods his head, yes). All right, now the next one.

In this exchange, the children's methods for solving problems are the topic of the discussion. As a consequence, the students' responses do not need to conform to a particular predetermined procedure and what they say is their own. This provides an opportunity for individual children to monitor their thinking, reflect on their prior solution activity, and make necessary cognitive restructurings. Additionally, the interaction allows for the exchange of viewpoints and the presentation of a variety of methods. The teacher's role in the discussion is one of initiating responses, directing the turn-taking, and highlighting solution methods. She accepts the students' answers in a nonevaluative way and paraphrases their ideas, thus inviting them to listen and to let her know whether or not she is accurately reflecting their solution processes. The children are actively involved not only as speakers but also as listeners, as evidenced by Andrea's comment, "I did it a different way than John said."

As the teacher highlights Andrea's solution, she extends her explanation to include the notion of noticing relationships between the problems. She also communicates to Andrea that her method is valued and at the same time is suggesting to the others, in a subtle manner, that this is a more efficient way. In addition to highlighting what is relevant, she also repeats the children's responses not only to clarify their explanations but also to interactively constitute what is accepted as a basis for mathematical argumentation in this class. This is exemplified in the exchange with John, who, following the teacher's request for explaining his procedure, replies, "... we know that 39 plus 20 was 59." At this point, he is only offering a partial explanation of his method. The teacher, expecting him to complete his solution, responded, "... 39 plus 20, they knew was 59. Then what?" In this way, the teacher communicates to John that he has not yet finished his explanation and that it is necessary to provide additional evidence to complete his reasoning. Moreover, rather than simply supplying the rationale herself, she directs the discussion back to John for him to finish.

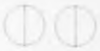
This discourse takes on the characteristics of a more genuine communication than the traditional school dialogue because the talk between the students and teacher is more equally distributed, or symmetrical. However, the discourse in this example is still an exchange that is for the most part between individual students and the teacher.

Situations for Discussion of Task Interpretation

In this example, the children have been talking about fractions for the past two days and have discussed the meaning of symbols such as 1/3, 1/6, and 7/8. In doing so, they have drawn circles and rectangles on the overhead and then discussed how to partition them into equal parts. On this, the third day, one child has asked a question about the existence of a fraction, 10/9. Partitioning a circle as 9/10 was feasible to him, but what about 10/9? As the discussion ensued and the children debated the possibility, they were unable to resolve the question. The teacher intervened and offered a suggestion, "What if I wanted you to show 5/4?" She drew two circles, wrote 5/4 on the overhead and asked, "Could you take these two circles and show me 5/4?" In response, Joel goes to the board and divides each circle in half.

Joel: Well ... a line through both of them.

Teacher: Okay. A line through both of them. (She draws a line through both circles.) Now what?



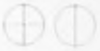
Joel: Then I think that should be it, because there are four pieces.

Mark: Um! I think I know why! Equals.

Alex: No. 5/4. It's possible with two things, but not with one.

Teacher: All right. Wait a minute (to Alex). Mark.

Mark: (Comes to the front and directs the teacher.) Make one with just four pieces and one with just one piece. (The teacher draws another line in the first circle.)



We can shade in four. This whole thing (points to the circle and the teacher shades it in) and one piece over here. (He points to the other circle and the teacher draws.) And 4 plus 1 equals 5! (Returns to his seat.)

Teacher: Is that a picture of 5/4?

Children: Yes, yes!

Following this discussion and after listening to another student's explanation for 5/4, the teacher continues the discussion by deciding to pose a problem in which fractions are presented in a discrete situation.

Teacher: If we have 20 kids and 25 apples, how are we going to split it up so everyone gets the same amount—equal pieces?

Sara: One.

Teacher: We can give everybody one, and what are we going to do with the other five apples?

Lisa: Throw them away.

Teacher: We're going to throw them away. Well, we can throw them away, but that's kind of wasteful.

John: Split them in half.

Teacher: Split them in half.

Children: Split them in fourths, split them in fourths.

Mark: Split them into 20 pieces!

Notably, this discussion is characterized by more active student participation and a dialogue that is more spontaneous. The teacher, in her role, is still guiding the discussion and influencing the students' responses by making suggestions, but in a more subtle manner.

Teacher: Right. Split them into 20 more pieces. So sometimes a fraction is not a whole thing or a group—it has (pause). It has extra pieces.

Alex: You split the apples into five pieces because. . .

Sally: (Interrupts) No, into fourths.

Teacher: Wait a minute. Shh (to rest of class who are all talking about the solutions). Okay (to Sally).

Sally: Five apples. 5 times 4 is 20.

Teacher: 5 times 4?

Sally: It would be fourths. Split the apples into fourths.

Teacher: We would split the apples into fourths. And so how much would everybody get?

Sally: One and a fourth.

Teacher: One and a fourth apples. Or we would get 5/4 (circles the 5/4 that was written on the overhead previously). Wouldn't we?

The discourse in this last segment illustrates how the teacher capitalizes on Sally's refutation of Alex's answer to create an opportunity for Sally to provide a rationale for her refutation. The teacher's actions indicate that although her role is to guide the discussion, it is the children's obligation to provide a justification for their thinking. Following Sally's comments, the teacher skillfully points out the connection of Sally's "one and a fourth" apples to the taken-as-shared meaning for 5/4 that evolved earlier in the discussion.

In this particular episode the ideas about fractions that were being discussed had not been predetermined by the teacher but rather had evolved during the course of the lesson. The children posed questions and offered answers to problems that arose as they talked about mathematics. The teacher, for her part, skillfully allowed the students to solve these problems for themselves as she has built from and extended their ideas. Thus, the particular connection that was made had not been anticipated by the teacher before she began the lesson but was developed as the teacher and students engaged in genuine communication in which the nature of the interaction was about making sense of mathematics.

Nonproductive Constructions

The previous episodes illustrate mathematical discourse in which children are engaged in an interactively constituted basis for mathematical activity that involves an exchange of viewpoints, which provides an opportunity for students to construct increasingly powerful conceptual operations. In this process of encouraging children to express their individual ideas, students often develop incorrect and incomplete constructions that reflect intermediate states in the process of construction and from which they will later form productive conceptualizations. However, children may also make some constructions that will be nonproductive and, if allowed to continue, will hinder their development (Lochhead, 1985; Streefland, 1988). Students in traditional classes also make nonproductive constructions, but in these classrooms they go unnoticed because children are only given the opportunity to offer single-word responses. The project teacher over the course of the year developed an awareness and a sensitivity for knowing when a construction a child made might be a sidetrack rather than a bridge to the next idea. The danger here is that if children retain nonproductive constructions, they may be left with incorrect notions from which they are unable to develop productive constructions. For the teacher this situation acts as a point of tension in that, on the one hand, she realizes that it is important to allow students opportunities for making errors in order for substantial conceptual learning to occur. Yet on the other hand, a real danger exists that an incorrect notion will constrain children's thinking in such a way that they will be unable to move beyond their present construction. The teacher, then, is constantly trying to maintain a balance between tolerating a child's error, recognizing that it may represent an initial understanding of a new construction, and identifying when it is nonproductive and there is a necessity to constrain it. In the following episode the teacher is confronted with a situation in which two children, Sally and Justin, are giving their answer to the following problem: "The bus to the airport has eight rows of seats. There are four seats in each row. This morning 23 people sat on the bus. How many seats were empty?"

Sally: We got 11.

Peter: I disagree.

Teacher: They came up with 11 seats. (She writes 11 under the 9 another pair had given earlier).

Justin: We knew that there was 8 rows of seats ...

Sally: (Interrupts) Well, we multiplied it.

Justin: So we thought 4 times 8 and got 32.

Sally: Because 8 two times is 16 and four times so, 16 plus 16 is 32.

Teacher: 16 and 16 makes 32. That's right. (She writes 32 seats).

At this point, even though Sally and Justin have given the incorrect answer of 11, their explanation and justification thus far is correct and indicates that their thinking has involved understanding repeated addition as multiplication.

Justin: Well there were 23 people on the bus that morning. There was 32 seats and that's a difference of 11.

Teacher: Okay. So they said altogether there were 32 seats on the bus, and 23 had people sitting in them (she writes 23 after 32 seats). So there are 11 seats with nobody sitting in them.

At this point, several other students disagree and begin offering their explanations for the original answer of 9, including one child who demonstrates counting on from 23, which has become the "rock bottom" (Davis & Hersch, 1981) procedure used in this class as the final verification of an answer. The teacher, listening, fully expects that one of the explanations that has been given by the other students has provided Sally and Justin an opportunity to reflect on their solution and so continues:

Teacher: Okay, Justin and Sally, what do you think?

Justin: We think it is still 11, because 32 take away 23. Well, 3 take away 2 is 1, and 2 take away 3 is 1.

With Justin's explanation, the teacher now realizes that the counterexamples and solution methods other students have created have not provided a conflict in their thinking. Furthermore, she also recognizes that for both Justin and Sally, the manner in which they are solving the problem reflects an operation on numbers that, if continued, will not lead to an understanding of two-digit subtraction involving decomposition. Thus, she continues the dialogue in a directive manner in an attempt to reduce the ambiguity in their thinking.

Teacher: All right, put up 2 fingers. (Justin puts up two fingers.) Now take away 3 of them. Can't do that, can you? So the way you are doing that is not working (pause). What is 23 and 10?

Justin: 33.

Teacher: 33, and we only have 32 seats and 23 people sitting in them (pause). Do you still agree with 11?

Justin: You get 9 out of it.

The teacher in this example proceeds in a directive manner to create conflict and to challenge their method by posing an alternative method for solving the problem. She asks, "What is 23 and 10?" and then points out that the bus has only 32 seats, implying that if 10 is too many, 11 will also be too large. She waits and then asks, "Do you still agree with 11?" This creates for Justin an opportunity to rethink his solution, to reconsider the validity of his procedure, and to make his final comment.

SUMMARY AND DISCUSSION

Teaching in a manner that is compatible with a constructivist's view of learning is a highly sophisticated form of practice in which the teacher acts to initiate and guide children's mathematical development in the ongoing complexity of the classroom. Notably, teaching in this manner is a radical change from traditional elementary school instruction and is more demanding than other forms of practice (Skemp, 1976; Thompson, 1984). From this perspective, teaching and learning is a reflexive activity that involves negotiation of mathematical meaning through a process in which a commitment to communicate is a mutual expectation (Cobb et al., in press). This form of pedagogy is highly compatible with changes advocated by the current reform in mathematics education. The recommendations for change in the way mathematics is currently taught emphasize the development of students' mathematical meanings (Wood et al., in press). This focus challenges the notion that teaching and learning consist of requiring students to replicate the teacher's activity and supports the contention that children's self-organization of their experience is essential (von Glasersfeld, 1988a). Such teaching involves an interactively constituted basis for learning mathematics in which discourse with a genuine intention to communicate is essential.

In particular, in developing a form of practice that is compatible with the recommendations, one aspect of the teacher's role is to guide the mutual construction of social norms, which are necessary for the development of an atmosphere conducive to solution processes that constitute a more sophisticated way of doing mathematics (Cobb et al., 1989). It also necessitates that teachers select instructional activities that create opportunities for children to engage in mathematical activity in which they can make personal constructions from their experience (Thompson, 1985) and to engage them in social interaction in which mathematical meaning is negotiated (Bauersfeld, 1988a). Additionally, the challenge for teachers is to find ways to facilitate and build on their student's ideas to encourage the construction of increasingly powerful conceptual operations. This requires listening to children's explanations and developing an understanding of the underlying conceptual operations that underscore children's thinking.

Moreover, if this form of practice is to be successful, two central issues must be considered. The first centers on the degree to which teachers understand their students' current level of thinking and the possible constructions they might develop. The second is the nature, quality, and extent of their own understanding and beliefs about the mathematics they are teaching (Ball, 1988; Ernest, 1989; Greer & Mangan, 1986; Noddings, 1990; Tirosh & Graeber (1989). If teachers need to develop an understanding both of children's processes and of the mathematics they are teaching, perhaps situations can be created in which opportunities exist for them to learn. In this regard, the classroom may offer a powerful site for teachers, as well as their students, to learn.

EPILOGUE: PURDUE PROBLEM-CENTERED MATHEMATICS PROJECT

In the years following the classroom teaching experiment, other second-grade teachers in the school district have requested to use what is now called the Problem Centered Mathematics Project (PCMP) in their classrooms as an alternative to textbook instruction. At this time, 20–25 second-grade teachers are involved, some of whom we are collaborating with as we continue to try to understand the nature of students' learning in classroom settings and the role teachers play in creating opportunities for their learning. Further, given the nature of the wider sociopolitical setting in which we collaborate with teachers, we found it necessary to conduct a quantitative comparison of project and nonproject students' arithmetical achievement, beliefs, and personal goals. The results have generally been encouraging from our point of view, particularly with regard to students' conceptual development and problem-solving capabilities in arithmetic, their perceptions of the reality of classroom life, and their personal goals as they engage in mathematical activity (Cobb, Wood, Yackel, Nicholls, et al., 1991; Nicholls, Cobb, Wood, Yackel, & Pastasnick, 1990). In addition, a follow-up teaching experiment was conducted in 1989–90 by Erna Yackel in one second-grade classroom in an inner-city school serving a minority population. This attempt to expand the theoretical perspectives to settings representing other populations is a necessary and important extension of our work in particular and the reform initiatives in mathematics education in general.

Finally, the wider sociopolitical setting has itself become an explicit object of investigation. This aspect of the project acknowledges that the process of reform in mathematics education is not limited to individual teachers and their students in classrooms but also relates to a broader level in which the school administration, school board, and wider community play a significant role. The section that follows, written by Deborah Dillon, an educational ethnographer, investigates the evolution of the project from an anthropological perspective in which she considers the project in light of the community specifically and the culture of schooling more generally.

Part 2

MEANING OF MATHEMATICS: COMMUNITY AND CULTURE

Chapter 7

THE WIDER SOCIAL CONTEXT OF INNOVATION IN MATHEMATICS EDUCATION

Deborah R. Dillon

In previous chapters of this monograph, the authors have written about children's learning in a classroom in which mathematical meaning is of central importance. These accounts allow us to see how recent recommendations for reform in mathematics education (NCTM, 1989, 1991) might look in action. After the initial classroom teaching experiment, the project continued to be used in a number of second-grade classrooms in the school district from 1987 through 1991. Despite the relatively successful efforts at educational reform at the classroom level, there have been ramifications of the innovation in the context of the wider society. These ramifications were not anticipated by the researchers and nearly resulted in the termination of the second-grade project. Thus, in the following chapter, I will present the evolution of the project by discussing the following: (a) the issue of innovation in mathematics education as it is embedded in a broader sociocultural and political context that includes the community, parents, school board members, and school administrators; (b) the influence of historical events in school districts on current reform efforts; (c) the influence of the perceptions held by parents, school board members, and school administrators on this innovative project; (d) the perspectives of researchers, teachers, and school principals about the project and how mathematics should be taught and learned and how these perspectives differed from those held by some parents and school board members; and (e) the influence of these differences on the process of innovative change in mathematics education.

BACKGROUND TO THE STUDY

Schools are systems embedded in, and reflective of, a society; moreover, they are systems shaped by past, current, and future events. Sarason (1990) warns, "As long as we uncritically accept the axiom and think of reform only in terms of altering classrooms and schools—what goes on in them—educational reform is doomed." Further, little is known about the power relationships within and outside school systems or how these relationships influence what occurs in school classrooms. Sarason (1971) illuminates part of the reason we know little about these broader contextual factors (e.g., social, cultural, and political) by observing that there is a "lack of systematic, comprehensive, and objective description of the natural history of the change process in the school." Finally,

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reports of educational reform efforts that go beyond a superficial understanding of the contexts in which education problems arise are scarce (Sarason, 1990). In short, it is clear that we must examine not only what occurs in classrooms and schools but how what occurs is influenced by broader contextual factors such as the social and cultural dimensions of society and individual communities. In the case of the second-grade project, it is important to understand these broader factors as they encompass the various perspectives individuals have about the purpose of mathematics instruction, the way they think mathematics should be taught and learned, and how these perspectives are shaped.

In this chapter, I will not only discuss the various contextual factors that influenced the near termination of the second-grade project, but I will also document the evolution of the project itself. This documentation can serve as a paradigm for those who want to learn more about the complexities associated with implementing innovative mathematics programs that are compatible with recent reform recommendations (e.g., NCTM, 1989, 1991; NRC, 1989). Indeed, mathematics educators are requesting such documentation. For example, the Final Report of the NCTM Task Force on Monitoring the Effects of the Standards (Schoen, Porter, & Gawronski, 1989) calls for the use of case-study research to document projects in which the standards are used in curriculum development. Specifically, these studies should describe, analyze, and interpret the inception of projects, key persons involved in projects, and steps in the process of change. Before discussing the evolution of the project, I will discuss, by drawing from relevant literature, the roles of culture, society, and politics in school settings.

THE ROLE OF CULTURE AND SOCIETY

Mathematics education researchers who have explicitly considered culture typically focus on teachers' abilities to adapt to the cultural backgrounds of their students in the course of instruction (Bishop, 1988; Gay & Cole 1967; Gerdes, 1988; Mellin-Olsen, 1987; Pinxten, 1983). For example, Bishop (1985) stated that "in order to understand social phenomena in classrooms one has to consider the classroom as a part of a much greater social 'framework.' " He further noted that "there [are] important social phenomena, customs, values, histories, etc. which play a strong shaping role in math education." Other researchers have also argued that we need to examine mathematics education at several levels, including cultural, societal, institutional, pedagogical, and individual. This treatment of culture, however, views the cultural level only as it affects what educators must do to understand the learner in the classroom and adapt instruction to the learner. Many scholars have not examined the cultural level as one composed of beliefs and attitudes held by community members that influence what these individuals believe should (and will allow to) occur in classrooms. Thus, we need to follow Chilcott's (1987) advice and examine these outside school factors:

> Each culture is unique ... the focus of school ethnography needs to be more diacritic, focusing on the sociocultural processes within and outside of the school that create the situations within the school. (p. 209)

Bishop, too, supports this diacritical notion by stating that the societal level, (a level to which he attributes great importance but notes it is not as well researched as the others) involves political and social issues—primarily the competition for power and control of the curriculum.

THE ROLE OF POLITICS

Reform efforts in schools are influenced by who has power and control of the curriculum. School board members and parents have strong views about curriculum and teaching methods. As Bishop relates, the societal level can incite struggles for power with regard to who determines what is taught, how it is taught, and how it is assessed. If the school and societal perspectives on how mathematics is taught and learned differ greatly, moments of conflict and power struggles arise. Popkewitz (1988) supports this claim, stating, "Classroom practice reflects both school traditions and sociocultural values, although they do not always coincide and can involve conflict." The politics of mathematics education, which focus on a difference in community members' sociocultural beliefs about what should be learned and how it should be learned, has not been examined. As Warren (1983) writes,

> By its very nature public schooling proceeds within a political and ideological context. Its community role as a primary socialization agency and its status as a state supported institution ensure that political import will be associated with educational activities. (p. 125)

Warren emphasizes that: "The assessment of contextual factors and of the interactions/interrelationships across contexts is viewed as requisite to program implementation and maintenance."

Popkewitz (1988) supports the need for research of this nature when he states, "The social patterns of school conduct are not neutral but related to the larger social and cultural differentiation that exists in our societies." He contends that when introducing innovation in mathematics education we must consider "the complexities of curriculum in an ongoing, cultural world in which there are unequal social relations and different interests." The perspectives presented above point to the need to understand the multiple layers of complexity influencing what occurs in school settings.

PURPOSE OF THE STUDY

The purpose of my study was to document the process of change experienced by teachers, researchers, and others who participated in the second-grade mathematics project. I also wanted to understand how this process was interpreted by the teachers and researchers and how it was influenced by members of the community, school board, and administration. The following research questions guided the study:

1. What happens when university researchers and public school teachers collaborate to develop an innovative form of educational practice in mathematics?

2. What role does the community culture, especially the beliefs and values held by its members regarding school in general and mathematics in particular, play in influencing what occurs in mathematics classrooms?

3. What role do parents and school board members play in informing and developing school district instructional policy, and how do they influence the process of educational change in mathematics?

THEORETICAL FRAMEWORKS AND METHODOLOGY

To facilitate the study of the historical, cultural, social, and political influences on the second-grade mathematics project, I adopted a holistic, anthropological perspective as recommended by Crow, Levine, & Nager (1989) and Eisenhart (1988). This perspective allowed me to understand the role of culture as it framed participants' constructed meaning about events within the community and the school system. I also adopted Geertz's (1973) semiotic concept of culture: "Culture is not a power, something to which social behaviors, institutions, or processes can be causally attributed; it is a context, something within which they can be intelligibly … described." The holistic perspective and definition of culture I adopted enabled me to better understand various participants' beliefs, power relationships, sense of control, and voice as they were shaped within cultural contexts.

Within this framework I used the methodology of ethnography. This methodology allowed me to intensively observe and interview groups of people in order to understand their culture—their beliefs, values, and taken-as-shared assumptions (Dillon, 1989). The theoretical underpinning of the study was holistic ethnography; this perspective allowed me to understand a community by describing the beliefs and practices of constituent groups and then showing how the perspectives of the various group members are interpreted within the context of the culture of the whole community. As Ogbu (1981) states, "The ethnographer must develop more of a holistic perspective that focuses upon the interdependency of variables affecting the school—upon structural causes of change within the organization.…"

Additionally, the study was undergirded by the theoretical perspective of symbolic interactionism (Blumer, 1969), based on the tenet that meaning for individuals is derived through the social interaction they have with others. Symbolic interactionism is important in this study because it allowed me to understand how individuals viewed what is taught in schools and how teachers teach and students learn, as these issues were discussed in meetings and through letters and memos. It should also be noted that there is a broad philosophical compatibility between the perspectives and theoretical frameworks that guided this study and those guiding the initial second-grade classroom project.

Setting, Participants, Researcher's Role

The setting was the school district where the project was being conducted. In fact, the near termination of the project precipitated my interest in conducting a study in this setting to understand what had transpired before and after the near termination. The participants in the study were the teachers, administrators, parents, and school board members from the school district and the researchers involved in the project. Because so many individuals were involved with the project, participants from each of the various groups were selected to serve as key informants. My criteria for selecting participants was to gather data from individuals who held varying viewpoints—those supportive and not supportive of the project. This selection procedure would ensure that multiple perspectives, representing the continuum of beliefs that existed in the setting, were obtained. (See earlier chapters for a more detailed description of the setting and the participants involved in the study).

I assumed the role of participant observer as I collected and analyzed various forms of data; however, I remained more of an observer than a participant. To prepare as a participant observer in this setting, I relied both on my former experiences as an elementary teacher and on my current experiences as a professor of education; my background helped me relate to the setting and the participants in the study. However, I did not have an in-depth understanding of the project or how it actually operated in the school setting aside from visiting a classroom and reading various research reports about the project. Further, I did not involve myself with the project as a member of the research team, nor did I become part of the teacher team or a part of the community. Rather, I attended meetings where the researchers met with the teachers and meetings where the teachers met with the school board expressly without the researchers. Throughout the study I found that working to maintain a somewhat detached yet involved role allowed me to gather more perspectives on the events that occurred. However, this unobtrusive role was not easy to maintain. Several times I found myself in a position in which the teachers requested my advice in their attempts to retain the project. In addition, the researchers were inherently curious and wanted information about what happened at meetings they were not privy to. Yet if I related information to either group of individuals, my intervention could have changed the naturally occurring events of the very process I wanted to understand.

It is also important to note here that because of the political nature of the data collected and of my relationship with the participants and because schools and people are easily identifiable when described in detail, I have opted not to use the participants' names or the name of the district in this chapter.

Data Collection

Even though many of the events related to the second-grade mathematics project had already occurred when I began the study, as an ethnographer I tried to reconstruct events with participants through interview and document analysis procedures (e.g., Dwyer, Smith, Prunty, & Kleine, 1987). Data collection began in the fall of 1988, two years after the classroom teaching experiment, with primary data being collected in the form of intensive, structured interviews (Patton, 1980) with the following persons: (a) the researchers, (b) classroom teachers involved in the project, and (c) principals involved with the mathematics project. These interviews were audiotaped and then transcribed. Secondary data were collected in the form of field notes written during or after meetings with the researchers and teachers and after I attended school board meetings. If feasible, these meetings were also audiotaped for later transcription. Other secondary data in the form of artifacts were also collected. These included parent surveys and letters; all documents produced by and often circulated between researchers, teachers, administrators, and school board members; and materials distributed to members of the community during the course of the project. Historical data about the community were also collected in the form of newspaper clippings and papers written about former school events. Finally, a researcher journal was kept that contained my reflections on my role and the events occurring around me during the process of my research.

Data Analysis

Primary sources were analyzed with analytic induction methods (Erickson, 1986) or pattern-generating procedures including within- and across-case analyses; secondary data were analyzed with document analysis procedures (Patton, 1990). First, the analysis centered on generating a timeline of events to facilitate an understanding of the evolution of the second-grade mathematics project from its inception (1985) through the near termination of the project (1988) and afterwards (1989–91). The timeline was constructed by reading and rereading field notes, various meeting notes, interview transcripts, and documents to search for patterns in the participants' perspectives on the project and the events and people involved in the project's inception and evolution. Within- and across-case analyses were conducted with the same data sources to search for patterns in the participants' perspectives concerning (a) how they felt about the project, including their beliefs about mathematics and the teaching of mathematics, and (b) their perceptions of the evolution of the project and why it evolved as it did. Comparisons and contrasts among participants, based on the roles they held within the project, were generated from this second analysis. Points of difference were probed further to understand their origins. The analyses of perspectives and the timeline were then combined in the interpretive stage of analysis.

In the next section of this chapter I will present the results of my analysis and interpretation via the analysis of the events that occurred over the course of the

1988–89 school year. The analysis is couched in the historical context that preceded and surrounded these events. I will also present the participants' perspectives on the second-grade project.

ASSERTIONS, ANALYSIS, AND DISCUSSION

Five major assertions were generated from the data analysis and interpretation process, forming the basis for the results and discussion. These assertions are as follows:

1. The different groups represented in the community—their values and beliefs about education and their propensity for getting involved in school issues (particularly school reform)—are important when researchers and school personnel collaborate to affect educational reform.
2. As the project grew from one to several classrooms it became more public or noticed within the community; with this notoriety several groups of people became involved in and influenced the evolution of the project. Thus, when and how individuals learn about reform efforts may affect their acceptance or rejection of these efforts.
3. The various groups of individuals involved in the project hold varying beliefs and values about schooling in general and mathematics in particular. These beliefs shaped what individuals found acceptable or unacceptable in project classrooms, including what should be taught in mathematics classes and why, how mathematics should be taught and learned, and how learning should be assessed.
4. Individuals from these different groups (e.g., school board members) assumed or were assigned various roles throughout the evolution of the project. The roles people assumed were often shaped by issues that emerged during the evolution of the project. These issues included various individuals' desire for power, control, and voice in school system matters and their relationships with others within the system. Specifically, when particular individuals involved with the project were assigned or assumed power, they also developed a strong voice, allowing them to influence and control people and events.
5. Communication among individuals involved in the project influenced the evolution of the project and the acceptance or rejection of project goals. Communication, or lack thereof, was also used as a means of wielding power over people and the project.

In the remainder of this chapter these assertions, generated from the data, will be explored and supported with incidents drawn from data sources. First, key historical events will be presented to provide a backdrop or context for the second-grade project. Second, the timeline of events surrounding the project will be detailed. Third, I will present the perspectives of each group of individuals involved in the project. Finally, I will compare and contrast the perspectives of all participants to show how one participant's perspective, role, and beliefs influence whether an innovative project is accepted or rejected.

Key Events in the History of the School System

In 1985, an event occurred in the district that seemed to change many individuals' perspectives about the role of educators and parents in making curricular decisions. Moreover, this event influenced decisions in general about what content should be taught in school, how it should be taught, and who makes such decisions within the school district.

The event began with the possible adoption of a new curriculum that dealt with values education, compiled by a committee made up of parents, community leaders, teachers, and administrators. (Several of the words that were used to describe the program are highlighted below in boldface type. In an attempt to foreshadow, I want to interject that these highlighted words were used unknowingly several years later by the second-grade project researchers.) The values curriculum consisted of **activities** in which students often read about a scenario and then **worked in groups** or **participated in whole-class discussions** to decide what they might do in various hypothetical situations and why. Education professors from the university supported the curriculum, stating that children needed to learn how to think through ethical problems to formulate their own moral codes. **Discussion** and **problem solving** undergirded the instructional approach along with the notion that **there are no wrong or right responses**; rather, what is important is **choice** and **giving a rationale for your decision**.

The chain of events surrounding the demise of the values curriculum was precipitated by one man from the community, also a member of the adoption committee, who denounced the curriculum after it was adopted. He criticized the curriculum, stating that it was "dangerous" and put forth by "humanists and behavioral scientists who don't want our children to learn the difference between right and wrong." His public denouncement, printed in the local newspaper, spurred many parents to action. One parent, who had previously led a campaign to "successfully eliminate fluoridation of H_2O at her children's school," (Butler, 1985, p. 33) spoke out against values clarification being taught in the schools. This parent was characterized in the newspaper as follows: "Armed with articles from newspapers and educational journals ... [this parent] meets parents and relates their children are being numbed to moral standards" (p. 33). After receiving a lot of press, this parent commented,

> I'm not a rabble-rouser and I'm not a narrow fundamentalist by any stretch of the imagination ... I'm simply sharing what I feel with parents.

Differences in beliefs about the values curriculum escalated into a full-scale conflict between parents, educators, students, and community leaders. Various arguments were presented in the papers and at public meetings by concerned parents and others; eventually a parent group was formed. Parents' arguments ranged from a concern that humanists had developed techniques (via the values clarification curriculum) that could gradually change a child's conscience, personality, values, and behaviors, to a concern for the time spent on this new curriculum that was being taken away from **learning the basics**.

The argument pertaining to the basics is exemplified in the following parent's comments:

> I question [the] curriculum when [the state] ranks so low in national scores ... there was a decline in test scores—why? Because school systems are not performing their basic responsibility of teaching academics.

This same individual claimed that high school valedictorians were entering college and receiving F's because "the school spent valuable time on values education instead of teaching the basics" (parent comment, 11/85).

Other comments about this curriculum hint at parents' beliefs about mathematics instruction, specifically a belief in the need for the teaching of the basic facts:

> Only after principles are ingrained can decisions and choices be properly made. We do not ask students to solve a complex math problem by selecting from several methods or choices without first spending a lot of time on the principles that must be applied each and every time a problem is encountered. (parent comment, 11/13/85)

Parents leading the fight against the values clarification curriculum were careful not to alienate teachers. However, some comments sent an implicit message about how some parents viewed teachers' roles:

> We're investigating a program, not the teachers or [school district]. We love and appreciate our teachers—it's the curriculum that we strongly oppose ... I don't have any problem when techniques [to teach values] are being used correctly but why should I allow it [the teaching of values] when I don't know how the teacher's going to use it [the techniques]? (newspaper article, 11/85)

In the end, the values education curriculum was rejected. Teachers, university professors, parents, and students were all affected by the events that had transpired. Not long after the values curriculum was rejected a new school board was elected, and one of the new members was a parent who had been outspoken against the curriculum.

Summary of Events

An understanding of these past incidents foreshadows the subsequent events associated with the second-grade project. First, the events that occurred in this district are strikingly similar to occurrences that have unfolded in other school districts. Conflicts usually revolve around differences between school personnel and the broader community members' values and beliefs concerning what should be taught in schools. Often, differences in opinion result in censorship of the curriculum by members of the community. Because of these conflicts, the power to determine the curriculum shifts from school personnel to community leaders. Further, these conflicts often catapult community leaders into positions of power such as that embodied in a school board seat (cf. Moffett, 1988). As a result of these shifts in power, trust between community members and school personnel breaks down, causing teachers and administrators to avoid taking risks in the areas of innovative curriculum or teaching methods.

In the school district under study, the shifts in power occurred as a result of the events that transpired during the innovative values curriculum discussion. First, the superintendent's influence with the school board and community may have been diminished because the values curriculum he strongly supported was rejected. Many teachers, community members, and university professors who supported the curriculum also lost much of their voice and influence in school matters. In contrast, parents who led the fight against the curriculum suddenly found that they had a powerful voice in affecting change. This voice reflected the values and beliefs held by many parents: learning the basics is crucial, test scores are important indicators for informing parents about whether their children are learning the basics, and teachers may not be able to teach a curriculum that does not have explicit plans and lessons prepared for them. This voice and the accompanying publicity garnered throughout the conflict was enough to give one outspoken parent the edge to win a school board seat. Further, this parent and several others were determined to continue their activist agenda. In their role as school board members they carefully examined all new curricula and teachers' activities in order to provide the "best education possible for their children."

The second-grade project was initiated within this context. Clearly, in this setting, history does, in part, repeat itself. In the next portion of this chapter I will chronologically detail the evolution of the second-grade project. In Table 7.1, a timeline of the project is provided to give readers an overall framework within which to place key events and the persons involved at various stages in the project. The order of persons listed in Table 7.1 indicates who knew about the project and when they became aware of it; this order also indicates communication patterns between persons about the project (e.g., the researchers who developed the project communicated primarily with one administrator and one teacher; then principals learned about the project, followed by school board members and the broader community).

THE EVOLUTION OF THE SECOND-GRADE PROJECT

1985 Through Spring 1987

The project began during the 1985–86 school year with one teacher and two university researchers. It was conducted in one classroom, at one school, and initially consisted of the researchers interviewing children individually to understand their mathematical conceptual development. During the 1986–87 school year, the classroom teaching experiment was conducted with the same teacher and three researchers. During this year the researchers and others referred to the project as a teaching experiment. Parent meetings were held the first week of school, wherein the teacher and the researchers explained the project, and eight weeks later, at which time the teacher presented a math lesson by modeling instructional strategies with children who attended the meeting with their parents.

At the end of the school year the researchers anticipated that the project would expand to include other teachers and schools. Although this had been initially agreed on with the school district, the researchers set up a meeting to confirm the plan with the assistant superintendent. In this meeting they learned that the assistant superintendent viewed the project as an experiment; he believed that once data were collected, then the project was finished. The researchers argued that this next phase was not an experiment or research; rather it was implementation, curriculum improvement, and teacher development. The assistant superintendent stated that he would need to consult with the principals of the seven elementary schools for their reaction, and when he did, three out of seven principals expressed an interest in the project.

In sum, the second-grade project began at the grass roots level in one classroom. It met with the approval of upper-level administrators, one principal, one teacher, and the parents of students in the class (see Table 7.1). During the first year of the project, there was some level of control over what happened with the project because it was limited to one school, one teacher, and one set of students and parents. As the project was targeted for use in six other schools during the 1987–88 school year, it became more public and came under the scrutiny of more people. At the same time, however, unbeknownst to the researchers, no formal or informal measures were taken by the administration to inform the broader public (i.e., parents and school board members) about the existence of the project.

1987–88

During the summer of 1987 a five-day institute was held for teachers who volunteered to learn more about the project. At the end of these sessions, 18 teachers expressed interest in participating in the project during the 1987–88 school year. These teachers taught with the second-grade curriculum throughout the 1987-88 school year, supported by the researchers and the original project teacher. The principals took advantage of opportunities to observe in project classrooms and were notably impressed with what they saw. The assistant superintendent stated that he was receiving positive comments from principals and teachers, and in February 1988 he signed a letter of agreement to allow the district to continue as a cooperating school system for the researchers, who were seeking new federal funding to extend the project to third grade.

In the spring of 1988, two events occurred that greatly affected the acceptance of the project. First, a mandated state competency test was administered for the first time in all school districts. This state-legislated test was quickly incorporated in the schools with little warning. As such, it came as a surprise to the researchers and school personnel. The purpose of the test was to assess whether students had learned basic skills in content areas—skills that enabled them to progress to the next grade level. The mathematics section of the test consisted of

Table 7.1
Timeline and Persons Involved in the Second-Grade Project

Researchers	Researchers Principal Teacher (1)	Researchers Principals Teachers (18)	Researchers Administrators Principals School board members Parents	Researchers Administrators Teachers School board members Parents
1985–1986	1986–1987	1987–1988	1988–1989	1989–1990
Interviews	Teaching experiment in one classroom	Pilot Study: expansion of second-grade project to 18 classrooms	School board members question project; almost terminate it on several occasions. School board member proposed experiment (project, modified, and traditional mathematics classrooms)	School board's experiment put into effect in the schools: (project, modified, and traditional classrooms)

Note. Communication tended to occur between the researchers, principals, teachers, and those parents who attended project presentations. The researchers also communicated with some school administrators. However, lines of communication were not developed to include school board members and the broader community, since it was assumed that administrators were sharing information about the project with these groups.

two parts, a Computation subtest and a Concepts and Application subtest. Along with assessing grade-level progress, the results of the test were to be used to identify those children who needed summer remediation or grade retention. It was also known that these test results would be used as the basis for school performance awards and funding by the state. Clearly, this test, with an emphasis on computation and basic skills, was in conflict with the theoretical foundation of the second-grade project, which emphasized concepts, reasoning skills, and more than a single solution to a problem. Second, new board members were elected in May who represented the community group who had opposed the values curriculum and who wanted a return to "learning the basics."

1988–89

Early in the fall, the researchers had a working session with the project teachers, who now numbered 22. The teachers stated that their respective principals had just returned from a meeting where they learned that the project was in trouble and that the school administration might be "pulling the plug on it."

At a school board meeting the day before, one board member asked the assistant superintendent several questions about the project. Two of the new school board members had received phone calls from five parents who were concerned because they did not understand the mathematics instruction in project classrooms. An additional parent had registered a complaint because her son had failed the state competency test and was retained. The parent blamed her child's failure on the innovative mathematics project. This questioning session at the board meeting resulted in the school board president confronting the assistant superintendent with the charge, "If you put a program in our schools that's damaging our kids you've done wrong." Board members requested a full description of the project, to be shared at the next meeting.

The assistant superintendent decided that it might be useful to eliminate the project, and thus he sought the elementary principals' support. However, the principals refused to comply. After meeting resistance from the principals, the assistant superintendent met with the researchers and informed them of the board members' concerns about the project. The researchers provided him with statistical analyses of the state test data, results from an additional arithmetic test, and data from a Beliefs and Motivation survey, which supported the benefits of the project. These data were used at the next board meeting, at which the researchers had been asked to make a presentation. The researchers gathered additional information for the meeting from project teachers. The information included letters from parents, the results of a survey about the project sent out to parents from one school, and the results of a survey sent to all second-grade teachers. Teachers also helped by inviting school board members to visit their classrooms to learn about the project first-hand.

In an analysis of the events that occurred thus far in the 1988–89 school year, several themes emerged. The first theme centers on the lack of communication between the researchers, school personnel, and board members about the project. In addition, very few people other than those directly involved in the project understood the curriculum or instruction used within the second-grade classrooms. The second theme that emerges is a political one. Curiously, the assistant superintendent, after supporting the project for two years, suddenly wanted no part of it. However, interviews with school personnel revealed that the assistant superintendent hoped to become the next superintendent; thus, he needed to maintain his credibility and power with the board. Because he was unprepared to address questions about the project, his initial response was to eliminate the project. Perhaps he believed that this would allow him to regain his credibility with the board. However, the principals and teachers convinced him to support the project.

1988–89, Continued

The nine board members, the assistant superintendent, other administrators, and the researchers attended the next school board meeting. Several project teachers and a principal were in the audience. During the meeting the assistant superintendent told board members that he would begin by answering the question, How did we get involved in the project? He stated that the second-grade project "is not an experiment, not a dissertation study; it is work done by serious, competent math educators who are supported by a lot of grant money." Second, he stated that the "program is conceptually sound and consistent with [district] curriculum and the state's proficiencies." He explained that participation by teachers is voluntary, that principals and parents are informed about the program, and that an evaluation component is part of the program. He ended by stating, "The program has attracted international attention to the [school district]."

One of the researchers spoke after the assistant superintendent, talking about the philosophy underlying the instructional approach and informing everyone that the project was compatible with NCTM standards; this researcher also noted that the project was recognized internationally as a good program. The researcher discussed how the project was grounded in 10 years of research (case study research, primarily). He stated that the goal was to "develop an exemplary program." Next the researcher discussed how the materials and teachers' actions work to "develop problem solving strategies in students—which are more important than mere memorization of facts and rules." He continued, "We want to educate students for the 21st century—basic skills are not enough." The researcher concluded by relating how project students also develop persistence, cooperation, and responsibility for themselves. The state competency test scores were then discussed, including statistical data revealing that the project students did significantly better on the Concepts and Applications subtest than nonproject students and achieved equally on the Computation subtest.

After the researchers concluded their presentation, board members turned to the teachers in the audience and asked them to comment on the program. Several project teachers commented positively about the project, sharing what they had learned and how excited they were about their students' learning. School board members then asked the researchers a series of questions that revealed their concerns with the second-grade project. These concerns centered on the use of cooperative learning versus working alone, the lack of communication with parents about the project, and the lack of emphasis on basic mathematical skills within the program.

The following excerpt from the meeting illustrates these concerns:

Member 1: Why are the students paired?

[The researcher discusses cooperative learning and how it is beneficial to students' learning.]

Member 2: I was in class today and they change partners?

[The researcher explains that you want to change partners. The researcher adds that the 40 parent questionnaires sent out by one of the principals all came back positive; the teacher evaluations of the program were positive, too.]

Member 3: I was the one that raised the questions last time. I had two kids in school. I served on the exemplary schools committee. I'm here to ask questions. I've had more than six parents concerned this year—one from last year, I think. The parents don't know what's going on. The partners thing—if kids are not into the partnership well, they have to look out for the kids who don't work well in partners. I'm not sure kids should be responsible for others—if they always work in partners. Also [one elementary school's] scores were at the 44th percentile and [another elementary school's] scores were 88.

[The researcher discusses the low test scores and suggests that the problem was with the researchers' supervision of teachers—the problem was not with what the teachers were doing. One teacher in the audience added that the students only work part of the time in pairs. The researcher concluded with "We want to make sure that one kid doesn't do all the work."]

Member 3: One parent complaint was a parent with a very bright child.

[The researcher states that the parent should talk to the classroom teacher.]

Member 3: Bank tellers and mathematicians work alone, not together.

[The researcher states that actually a lot of mathematicians work together.]

Member 3: I'm talking about the person at McDonalds—they won't have a partner to help make change.

[The researcher concedes that this is a good point.]

Member 3: I suggest that parents who haven't been talked to, are. Some don't know anything about this program.

[The researcher comments that the board member is alerting them to a problem they didn't know existed—the parent meetings that have been held and that parents have attended have gone well.]

Member 3: (Looks around at other board members.) What are we going to do? It's [the project] a problem we can deal with internally.

Member 1: How well would your program have succeeded if you hadn't had in-service for the teachers?

[The researcher explains that it wouldn't have succeeded.]

Member 1: I have another question—based on your answers: Is this an experiment?

[The researcher explains that yes, it was an experiment the first year but that it isn't now—last year it was a pilot program.]

Member 1: How far are you from writing a textbook?

[The researcher explains, "Never. We know we won't improve the quality of mathematics by merely writing a textbook."]

Member 1: (looking perplexed) Where is this [program] being used outside our system?

[The researcher assures this board member that several other schools are using the program.]

Member 4: I've been contacted by parents also. Being here tonight, I see the philosophy. But I can see why parents have questions—it's very confusing. I mean worksheets with answers but no problems? Communication is essential.

Member 3: Parents say their kids are weak in the basic skills. When do the kids get help with this? As a school board we need to know that in the future these kids won't be struggling with the basic skills.

[The assistant superintendent joins the dialogue and states that the program was modified—it covers the basic facts.]

Member 3: Are facts checked on a regular basis by classroom teachers? One of the children who flunked the ISTEP [Indiana State Wide Testing for Educational Progress]—his poor math skills weren't reflected on the report card—his parents had no idea the kid was doing that badly. See—the kids are not grasping these basic facts.

Member 1: What is your program called again?

(School Board Meeting, 9/26/88)

[Note. The meeting continued with other agenda items—no decision was reached about the second-grade project.]

At the request of two board members, the researchers later answered additional questions about the project. These two board members shared the concerns parents had expressed to them including (a) university people experimenting with their children, (b) unusual worksheets being sent home, and (c) whether students were being assessed throughout the semester with the new program. Both board members agreed that students need problem-solving strategies but felt they also needed basic skills. One member again commented on the issue of cooperative learning, stating her beliefs that students need to learn to work independently in order to survive and be competitive in our society. This member added that cooperative learning taught values of sharing and helping and not those of independence and individuality.

In analyzing the presentation made by the researchers and the questions asked by board members, clearly a difference existed between what each of these groups saw as the benefits and difficulties associated with the project. The researchers focused on the benefits of **problem solving,** whereas school board members emphasized the need for **basic skills**. Board members were concerned that students be able to work **independently**, whereas researchers wanted **cooperative** learning. Several other board members seemed to believe that a new curriculum, devoid of a textbook and dependent on in-service training, must be scrutinized closely.

1988–89, Continued

In October the researchers wrote a letter to the assistant superintendent reflecting on the 26 September 1988 school board meeting and various members' comments. To address board members' concerns, the researchers stated that a series of pages called "My Work" would be given to students to complete individually and take home every 3 to 4 weeks. Further, the researchers promised to work with the teachers to ensure better parent communication. In addition, a proposal was attached asking that the project be expanded into third grade now that the funding had been approved by a federal agency. Within the body of the letter one of the researchers commented on what he perceived to be misconceptions held by board members about the program. The researcher stated that he and the board held the same belief—a desire for quality mathematics education for children. He discussed how project teachers were not attempting to teach values, but that "all teaching involves a visible and hidden curriculum … and students unavoidably develop forms of motivations and beliefs … and these can be interpreted in terms of values … but that the values learned in this mathematics program appear to be completely consistent with the values of the wider community." The assistant superintendent sent a copy of the researchers' letter to each board member with a cover memo requesting that the board permit the expansion and continuation of the project during the 1989–90 school year.

At a December 1988 school board meeting (the researchers and teachers were not in attendance), one board member called attention to the state test scores at one school, focusing on the project students' scores, which were lower than those at the other schools. Further, this member commented on the values issue associated with the project (based on the wording of the researchers' letter). This member proposed an experiment for the next school year in which the six participating schools would be divided into three groups—two schools to follow the established project instructional approach, two to teach the traditional textbook curriculum, and two to use a modified approach (which included the project approach but without the cooperative learning). The board approved the idea.

On 14 February 1989 a meeting between the president of the school board, the board member who instigated the experiment, the assistant superintendent, and the researchers was held to discuss and resolve the future of the project including the proposed expansion into the third grade. During this meeting the following issues were resolved: (a) the second-grade project could continue in "its current form" until the end of the 1988–89 school year; (b) the second-grade project could continue through the 1989–90 school year "intact" in two schools but modified in two others; (c) the school district would "support the third-grade project"; and (d) a group of teachers would "be organized to review second-grade mathematics instruction.... Using the traditional mathematics program and certain components of the second-grade program," this group of teachers would "develop materials ... and activities ... to enhance the second-grade mathematics program." The agreement ended with this statement: "The representatives of both the Board of Education and University agree on this process."[1]

In reflecting on the events that transpired over the 1988–89 school year, it is clear that the power and voice shifts that occurred early in the year between the researchers, school personnel, and members of the board escalated during the school year. Power was held by the board members; thus, the researchers, administration, principals, teachers, and most parents were not part of the decision-making process regarding the second-grade project. State competency test scores, the issue of cooperative versus independent learning, problem solving versus basic skills, and six parent complaints were the basis for the arguments presented by board members against the project. The researchers, not aware of several community members' strong beliefs about mathematics and how it should be taught and learned, quickly found out how these perspectives differed from their own. Exacerbating these differences in values and beliefs about schooling was the language inadvertently used by the researchers. Specifically, the choice of words used to describe the project (activities, working in groups, class discussions, problem solving, choice, and accepting right and wrong answers) influenced board members' beliefs that this program was not appropriate for the students in their district. And board members believed that they, as insiders in the community, knew better what the students in their district needed. In sum, a breakdown in communication occurred among all the parties involved with the project, with differences (a) in beliefs and values about what mathematics should be learned in school and how it should be learned and (b) in the language used by the researchers and the board members. All these factors contributed to the board's rejection of the researchers' program and the adoption of their own proposed experiment.

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[1] During the 1989–90 school year when the board's experimental program went into effect, several changes in the administration occurred (i.e., new superintendent, new assistant superintendent, and several new principals). In brief, all of the second-grade teachers (those in the textbook, modified, and project groups) organized to form a support group, working to communicate among themselves. They constructed a plan to convince the board to allow all of them to teach mathematics using the second-grade project in its original form. The teachers did not directly work with the researchers on this plan. Rather, they worked with new administrators in the superintendent's office and principals. At the end of the year, the results of the board's experiment, based on state competency test scores and the Beliefs and Motivation survey, indicated no differences between the modified and project classrooms. However, differences did exist between modified/project and traditional classrooms in favor of the modified/project. The new assistant superintendent analyzed the results and prepared a detailed report, which he presented to the school board. The teachers also presented at school board meetings throughout the year; they collected their own data on classroom learning and presented the results of their study to the board. As a result of these presentations, the school board voted to allow the second-grade teachers the choice to teach with either the project materials, other materials, or with the textbook. This story is detailed in Dillon, 1991.

A CLOSER LOOK AT THE PERSPECTIVES OF THE PARTICIPANTS

In order to make some sense out of the events presented in the previous section, I would like to take a closer look at the perspectives of the participants involved in this study. Such an analysis allows a better understanding about what occurred and why. Moreover, an analysis such as this might help us better understand how current NCTM recommendations for reform will be influenced and interpreted by the wider sociopolitical community.

The Researchers' Perspective

The **role** the researchers assumed, whether self-created or not, was that of outsider. Researchers are part of the university community—people who ask school administrators if they can do experimental research in the schools. In this study the researchers' goal was to create an innovative mathematics program and support teachers as they worked with the curriculum. This goal grew as more teachers became interested in improving the quality of mathematics in their classrooms. This goal was achieved, in part, by providing teachers with instructional activities and learning opportunities. The researchers, by virtue of developing the instructional activities, had a voice in what mathematics instruction would look like in the school district. Because they believed that a problem-centered approach was essential for children's learning of mathematics, they worked to garner support for the project, starting with teachers at the grass roots level. (However, starting at this level was not a conscious decision made by the researchers.) Through the process of in-service training for teachers and support as they made use of the activities, the researchers controlled, to an extent, the way the instructional activities were enacted. To fully realize their goal the researchers needed teachers who were willing to change what they and their students were currently doing during mathematics lessons. Further, the researchers had to convince principals and other administrators of the value of the project.

In expanding their "experiment" into a "pilot study," the researchers took the risk of jeopardizing the project without realizing it. When the project went beyond 1 classroom to 18, the researchers opened themselves up to the scrutiny of administrators, parents, and school board members. They assumed the additional responsibility for the learning and development of the 18 teachers during the 1987–88 school year and ultimately for the results of the mathematics sections on the new state competency test. Assuming this responsibility meant trusting the teachers to work with them in using the instructional materials to the best of their ability. It also meant trusting principals to support what these teachers were trying to accomplish. Thus, as the number of teachers, students, parents, and administrators involved in the project grew, the researchers had less control over the details pertaining to the project (see Table 7.1 for the individuals involved in each stage of the project).

Interestingly, there were no parent complaints logged with school board members during the initial years of the project. When parents did become concerned about their children's mathematics (during the third year), they did not call the researchers, teachers, or principals—they called school board members. This concern was coupled with the election of new school board members and the instigation of the state test. These events initiated a shift in power and control of the project. New school board members with an activist agenda did not understand the project; when they questioned the assistant superintendent, his lack of knowledge enhanced their belief that they and the parents they represented had not been sufficiently informed about the project. At the school board meeting when the researchers attempted to inform board members about the project, board members were more interested in hearing from teachers, principals, and parents about the project, appearing to find the comments of individuals within the district more credible than those of outsiders. In sum, the lack of appropriate means for communication among the various individuals involved, the differences in perspectives about mathematics, and the lack of trust between project personnel and school board members resulted in the near termination of the innovative curriculum.

The Teachers' Perspective

Many of the project teachers volunteered to participate in the second-grade project summer institute because they were not satisfied with their textbook-based instruction and considered themselves to be weak in mathematics. Project teachers conveyed that they decided to accept the "challenge" and work to change their mathematics instruction and their role as teachers. To do this, they assumed the role of "learner" in the earlier stages of the project and viewed the researchers and the project teacher as "the experts."

After attending the institute, teachers had the option to participate in the pilot program or to continue teaching in a traditional manner with a textbook. In a sense, they had a voice in the instructional materials that they elected to use. Those who chose to use the innovative curriculum had control over how these materials were used in their classroom. Initially, this control was mutually constituted between the researchers and the project teacher during visits to the classroom. The researchers trusted that the teachers would use the instructional activities and strategies in a manner compatible with the underlying tenets of the project. The teachers, too, trusted that the researchers would support them—that they would continue to be there when the teachers needed them. As the project teachers gained experience using the instructional activities and their understanding of children's thinking about mathematics grew, they became excited about what they were learning about themselves as teachers. The confidence and excitement they felt was manifested in a sort of power, a realization that they could be, and were, effective teachers (Dillon, 1991). Along with this power came a new role, one of "collaborator" with the researchers.

Despite their new collaborative role, when teachers heard that the project was in trouble they immediately went to the researchers, believing, in part, that they could intercede to resolve the conflict. The teachers believed that the experts would be able to inform the school board about the project and that the curriculum would be accepted. Some teachers, however, believed that it was a losing battle with the board from the beginning. These teachers based their beliefs on past events in the district's history (including the recent downfall of the values clarification curriculum). Nonetheless, project teachers attended school board meetings and voiced their enthusiasm about the project, adding anecdotes about the kinds of learning taking place in their classrooms and inviting board members to "come and see for themselves." Throughout the year, in the isolation of their classrooms, these teachers voiced frustration that an instructional approach that they believed in and worked so hard to develop could be taken away from them because, as one teacher commented, "a few parents complained" and "a school board member fears cooperative learning." Project teachers also voiced bitterness that some board members seemed to "gauge everything on one test score". The influence of a few voices (parents and board members) over so many voices (all the second-grade teachers and parents who liked the project but did not vocalize their support) was frustrating to teachers. Further, they discovered that some members of the board felt that they should determine what curriculum teachers were to use and how they should teach. In essence, they realized that board members did not consider teachers to be members of a profession.

The Administrators' Perspective

The upper administration (the assistant superintendent and the superintendent), although informed about the project, were not actually involved in it until the first crisis point. They signed the paperwork to allow the researchers to conduct the classroom teaching experiment and to expand the project as a pilot study, thus exerting their power as administrators. However, they had no control over the project. Reports from satisfied teachers and principals were the only updates they appeared to need. Board members' questions at a school board meeting caught the assistant superintendent unprepared, causing him to lose voice and power in his position as curriculum overseer.

The Parents' Perspective

A few parents, many of whom had not attended parent information meetings at the beginning of the school year, voiced concern when they noticed differences in the papers their children brought home. Their first concern was the lack of traditional textbook worksheets with grades on them. Because of this, parents stated that they had too few indicators of their children's learning or the progression of that learning. Further, parents expected the worksheets that did come home to look like traditional worksheets—full of addition and subtraction facts. Instead, the papers from project classrooms contained answers and no problems during the first weeks of school when problem-solving (TOPS—Techniques of Problem Solving) cards were used. When parents asked their children what these worksheets were all about the conversation resulted in confusion and a misunderstanding that the school was teaching "new math" again. Children also told their parents that they worked with partners on math problems; some children related that they liked their partners, and others stated that they disliked their partners.

Word spread among a few parents that the project was an experiment; thus, parents grew concerned that their children were subjects in a university research project that could potentially cause them to fall behind in math. At the end of the school year, state test scores were presented as an argument against the project by one mother who claimed that her son had failed and was repeating second grade because he was in a project math class.

Interestingly, positive comments outweighed negative comments. Parents who came to school and observed project lessons noted that they could see the quality of the program—"like the story problems." When one parent heard the type of discussion students engaged in she remarked, "Gee, I didn't know my child could think that way!" Parents were also pleased with their children's enthusiasm about mathematics as a result of the project. One parent reported that his child's "self-concept seemed to be enhanced" by the project, and another parent reported that she believed that her child's mathematics skills were "pretty advanced relative to other second graders."

In sum, most parents who were informed about the project supported it. Others did not seem to be aware that a new curriculum existed. Support for the project appeared to be secondary, however, to many parents' expectations about what they believed second-grade mathematics instruction should look like; these expectations were not totally fulfilled by the project. All in all, most parents trusted their child's teacher and the administrators who endorsed the program. Thus, it could be inferred that those parents who voiced their complaints directly to school board members, despite their extreme minority status, could still exert tremendous influence.

The School Board Members' Perspective

With parent complaints as a foundation for their concerns, board members began to question the validity of the project. When the researchers presented information at the September 1988 board meeting, some members had already gathered their own data. These individuals had investigated various aspects of the project by searching for articles and books comparing the merits of problem solving versus basic skills in mathematics programs. Board members also examined the corporation's state competency scores.

Thus, the issues that concerned board members were parents' lack of knowledge about the program and the cooperative learning aspect of the project. In addition, the board was concerned about the role researchers, as outsiders, had in determining how mathematics should be taught and learned. For example, there was a concern that the district would be dependent on the university for the curriculum used in mathematics instruction and for the in-service training on how to teach with the materials. Then there was the issue of the authenticity of the project curriculum; was it acceptable to other districts and were other districts using the program? Board members believed that an authentic curriculum is eventually articulated in a textbook, an accepted authority containing knowledge that has been deemed valuable and no longer experimental.

Another concern for board members was how students' learning was assessed and what the artifacts of this learning looked like. Some parents had voiced concerns about worksheets with answers but no problems taken home by children. The concern about unfamiliar artifacts alerted board members to a complementary concern of parents that their children were weak in the basic skills. The logical inference was that the project approach was not addressing the basic skills. By stating, "As a school board we need to know that in the future these kids won't be struggling with the basic skills," one board member reminded her peers of their responsibility to monitor new curriculum while also reinforcing the authority that the board had to reject curriculum perceived to be detrimental to students' learning. Further, questions pertaining to whether the "facts are checked on a regular basis by teachers" revealed board members' perception that their job was not only monitoring what is taught (basic skills) but scrutinizing how it is taught (monitoring teachers to ensure that they are responsible in carrying out their duties). Thus, the board later designed an experiment to take back control over what mathematics content would be taught and how it would be taught in second-grade classrooms.

Table 7.2 summarizes the various perspectives held by the participants in this study. In Table 7.2, I juxtaposed the various beliefs and values held by the participants. Of note are the issues considered important to each group of persons

Table 7.2
Varying Perspectives on Mathematics

Researchers	Project is congruent with NCTM standards; recognized nationally and internationally. Project is grounded in 10 years of research (case study primarily). Project helps students develop problem-solving strategies, which are more important than memorization of facts; basic skills aren't enough; also develops persistence, cooperation, responsibility, and motivation. Project provides in-service training for teachers, thus beginning change at the grassroots level; parents are involved, also. State competency test scores show project students' scores are higher on the concepts subtests (significantly) than textbook-based students.
Teachers	Concern for students' learning (what they learn and how); concern for students' motivation to learn mathematics. Frustration with concepts, format, and teaching ideas in traditional textbooks; concern with perceived inadequacies related to teaching mathematics; personally rewarding to use project. Strategies students learn spill over into other subjects. State competency test scores do not show what students know/understand.
Administrators	State competency test scores are important indicators of learning. Project program not an experiment; funded [Federal agency] and run by serious, competent math educators. Project consistent with school district curriculum and state proficiency guidelines. Project has attracted international attention to school district. Teachers pleased with program; principals believe program is good; parents like program.
Parents	Desire quality education for children. Want to see familiar signs of schoolwork such as traditional worksheets. Want ongoing indicators of children's progress and the nature of this progress. Want to understand curriculum being used; do not want children to be experimented on. Want timed tests and drill on basic facts in math class; state competency test must be passed. Want math to be fun; to build students' self-concepts.
School Board Members	Parents' concerns: —experimenting with children —no worksheets —unusual worksheets —partners —assessment problems Cooperative learning is not good; students must learn to work on their own. State competency test scores are not as high as they should be, in fact higher in nonproject school. Lack of emphasis on basic math skills in project. In-service training needed for teachers to be able to teach with project curriculum. Concern with what individual teachers are actually doing in their class rooms. Concern with lack of textbook—project looks like sheets of paper in a note book. Concern with the idea that only a few schools are using the project.

Note. Issues considered most important to each group determined the order of the listed perspectives. The participants' statements are taken directly from audiotapes, interviews, and documents.

with regard to mathematics teaching and learning, indicated by the order of the comments made by each group. In examining the comments we see that issues of primary concern to one group were apparently of lesser concern to other groups.

SUMMARY AND DISCUSSION

Insights and Issues

What happens when university researchers and teachers collaborate to develop an innovative form of educational practice in mathematics? It appears that reform programs, initiated at the grassroots level, can fail. Even though pedagogical change spurred by teachers may need to occur in this bottom-up fashion, support for innovation at other levels is essential to ensure long-term support for change. We have tended to look at innovation only as it occurs in classrooms and then perhaps within school systems. However, we have neglected to examine the influences of broader cultural, social, and political forces that shape what change occurs within schools and the nature of this change (cf. Sarason, 1990). Effective change initiatives must be examined and understood as they are embedded in schools as systems with the added sense of how these systems are situated in particular communities.

With this broader context in mind, how do we catalyze the process of change? First, it may be that an interactive, multiple-pronged approach to innovation is necessary. This process would include examining and understanding the histories and current status of school systems and the communities in which they are embedded. Within this broader context are the values and beliefs various groups hold concerning what content should be taught and how it should be taught. Power relationships among groups of individuals would need to be unearthed and hidden agendas exposed. From the story shared in this chapter, it appears that an insider (a member of the school community) is a key element in reform efforts. He or she can serve as a facilitator or broker between university personnel, school administrators, teachers, and community members. In fact, an individual like this emerged to help with levels of communication pertaining to the second-grade project during the 1989–90 school year (see Note 1).

Second, it is clear that parents and schools board members have the potential to influence and shape innovative change. Accordingly, researchers and teachers need to construct new ways of working collaboratively with these individuals. One way to begin is to hold sessions in which all participants feel they have important information to offer. Researchers and teachers can share information about current national trends in mathematics education. Community members can share what skills they believe their children need and why. All participants need to examine the influence of long-standing beliefs about teaching and learning in general and mathematics in particular, ultimately focusing on how these beliefs can influence reform. From this dialogue could come a working relationship among parents, community leaders, teachers, and university personnel, resulting in the construction of instructional materials and approaches, informed by parents' and educators' concerns yet grounded in current research knowledge. Further, opportunities for dialogue need to be planned over time, allowing researchers to talk with teachers, administrators, and parents about current research in education, while simultaneously allowing researchers to understand the culture of the community and gain valuable insights from parents about their children.

What role does the community culture, especially the beliefs and values held by its members regarding school in general and mathematics in particular, play in influencing what occurs in mathematics classrooms? The answer to this question is important in understanding the evolution of this project. A few parents voiced concerns about what students were learning in mathematics because the artifacts of learning—a textbook, worksheets, basic skills sheets, and timed tests—seemed to be missing. Parental concerns can cue researchers and teachers into the need to work with parents and community members to inform them of how these unfamiliar artifacts reflect students' learning and growth while concurrently reforming teaching and learning.

Standardized tests and competency tests—instruments that typically measure basic skills—influence both what is learned in mathematics classrooms and how it is learned. These tests appear to be viewed by parents and school board members as more valid indicators of students' learning than tests of students' conceptual understanding or students' problem-solving capabilities. Teachers' observations or the records they keep of students' learning are also not perceived as valid assessment tools. Further, parents and board members appear to be more concerned about high test scores than with students' attitudes toward mathematics or their motivation to learn. Thus, in this study the state competency test held a great deal of power over what content parents and the community believed teachers should or should not be teaching and how this content should be taught. The power of standardized tests is also reflected in the rewards associated with high scores, for example, monetary awards from the state and status for the community. Again, parents and researchers need to engage in dialogue to determine what these tests do or do not tell us about students' learning and how they may restrict important kinds of learning in schools.

What role do parents and school board members play in informing and developing school district instructional policy and how do they influence the process of educational change in mathematics? It appears that parents and community leaders can play a powerful and often unpredictable role in influencing educational change. Parents, in an attempt to help their own child, often neglect the total picture of teaching and learning and school reform efforts. Board members, in an attempt to represent parents' views, are often guided by the loudest voices rather than by the many voices of parents, teachers, students, and administrators. Further, when historical events have occurred (e.g., the values curriculum conflict) that raise the public's suspicion about what is taught and how learning occurs, then we should expect that new curricula will be closely scrutinized.

As I reread each participant group's perspectives (see Table 7.2), I am struck by the common concern voiced by all individuals who participated in the study—providing the best mathematics instruction for students. Interestingly, "what is best" for students depends on one's perspective. Further, should we only be concerned with students' learning, since this is only one part of what makes up the whole of schooling? And for whom do schools exist? Sarason (1990) notes that often the public believes that schools exist only for students. He argues that schools should exist coequally for students and educators. In other words, teachers' ongoing learning and development and their empowerment to make decisions concerning what occurs in their classrooms contributes to their job satisfaction and ultimately to the improvement of schooling. But is the general public ready for such an idea?

In reflecting on the second-grade project, the broader community and many individuals within the school system (e.g., some administrators) did not appear to be open to the idea of empowering teachers—allowing them to make decisions concerning what curriculum should be used or what teaching and learning should look like in their classrooms. For educational reform to occur in mathematics classrooms and beyond, it appears that all participants (students, teachers, researchers, administrators, parents, and community members) must rethink the purposes of schooling to include the needs and perspectives of all involved in the process of schooling. Perhaps through reflecting on cases such as the one described in this chapter, those involved will begin to consider the complexity of what occurs in schools, the obstacles to reform efforts, and ways we can work together to reconstitute schools into places where all can engage in meaningful learning.

Part 3

COMMENTARY

Chapter 8

FOCUS ON CHILDREN'S MATHEMATICAL LEARNING IN CLASSROOMS: IMPACT AND ISSUES

Martin Simon

In accepting the invitation to provide a commentary for this monograph, I took on the task of providing a view of the research as an observer from outside the project. It is always a challenge to add something of value to quality research that is articulately described. This chapter focuses on the theoretical underpinnings of the project described in Part 1. In Chapter 9 I will discuss Dillon's ethnography (Part 2), which describes mathematics reform in the context of the larger community. The multiple perspectives reported in this monograph—cognitive, sociological, and anthropological—afford us a unique opportunity not only to see the interaction between different domains but also to view the process of mathematics reform at different grain sizes.

The research described in Part 1 of the monograph contributes significantly to the mathematics education literature in several ways. First, the theoretical perspectives and constructs developed in the course of the study provide useful lenses for viewing mathematics classrooms and valuable frameworks for understanding the complex processes that characterize the classroom community. The work of Paul Cobb, Terry Wood, and Erna Yackel reflects a solid theoretical base. The researchers have brought together constructivism and symbolic interactionism to develop a useful social constructivist lens for looking at mathematics activity in classrooms.

A second level of contribution is made by the project's research methodology, particularly the constructivist teaching experiment (Cobb & Steffe, 1983), which the researchers have extended to be a tool for studying the mathematics classroom as a whole. And finally, the research results of this study contribute in important ways to knowledge in the field. The study provides us with a close-up look at an inquiry-mathematics classroom, both how it is established and the nature of the mathematical activities that transpire.

Following the discussion of the contributions the study has made, I have chosen to challenge some of the interpretations of data that appear in Part 1. My purpose here is not to insist on my interpretations but rather to use my own perspective, grounded in my research on mathematics teacher development, to broaden the discussion (Simon, 1989, 1991; Simon & Schifter, 1991).

In the third section of the chapter, I suggest some of the important work that could derive from this study. The extensions considered involve classroom studies that focus on the learning of particular mathematical content, classroom studies in upper grades and some of the issues that such work raises, and studies that relate teacher development (mathematical and pedagogical) with the teacher's role in constituting the classroom mathematics community.

CONTRIBUTIONS OF THE SOCIAL CONSTRUCTIVIST THEORETICAL PERSPECTIVE

The social constructivist perspective, which is fundamental to the research of Cobb, Wood, and Yackel, recognizes the role of the individual construction of knowledge, the social construction of knowledge, and the interaction between the two. Each contributes to and constrains the other. The cognitive schema of the individuals making up the group, in this case members of the second-grade classroom, contributes to and constrains the nature of the mathematical knowledge that is constructed by the classroom mathematics community. Likewise, the nature of the socially developed taken-as-shared knowledge and conventions for developing and evaluating new knowledge constrain the individuals' processes of meaning-making. Individuals find themselves part of a number of communities, often one embedded in the other, that enhance and constrain their development of knowledge. Thus, the classroom community is embedded in the larger school and district community, which is encircled in the local community, which is embedded in ever larger political communities. In addition, the classroom community is encircled in larger mathematical communities. Each of these communities influences the nature of the knowledge that is constructed.

An important contribution of the social constructivist perspective, as described in this volume, is the assertion that no analysis of students classroom learning and activity is complete without considering both a sociological and psychological perspective. It is only with this larger view of the classroom setting provided by a coordinated analysis that we begin to see a context for making sense of student and teacher activity in the mathematics classroom. Examining only the mathematical problem under consideration and the relevant understandings of the problem solvers causes us to miss a considerable amount of what is problematic in the situation (e.g., the "mathematics tradition of the classroom") that motivates the particular constructions of the individuals in the group. Individuals not only are negotiating mathematical problems, they are also attempting to be effective in a social situation (Balacheff, 1986). To be effective, the individual builds on the taken-as-shared knowledge of the group and interprets situations in light of his beliefs about how this classroom functions and the nature of mathematical activity. Likewise, focusing only on the social interaction and the construction of taken-as-shared meaning neglects the rich and varied set of mathematical schemes that individuals bring to bear on the problem at hand.

A second powerful contribution of the social constructivist perspective is its ability to elucidate developing notions of the nature of mathematics and mathematical activity in classrooms, notions that live in the intersection of the psychological and the sociological domains. A third strength of the perspective as developed and applied in this monograph is its usefulness for looking at classrooms that represent divergent mathematics traditions based on different epistemological stances. (See Cobb, Wood, Yackel, & McNeal, in press.)

CONTRIBUTIONS OF THE RESEARCH METHODOLOGY

The teaching-experiment methodology extended to the classroom is ideally suited for research of the type described in this volume. The methodology, which is based on constructivism, provides a unique context for understanding the development of mathematical knowledge and the constitution of a classroom mathematics tradition.

Constructivism assumes that humans are sense-making beings, that they have a natural drive to make sense of their experience. One can view mathematicians or people engaged in mathematical activity as trying to make sense of the mathematical world and to make mathematical sense of the world. The mathematics education researcher in a classroom teaching experiment is engaged in trying to make sense of the mathematical activity and social interaction observed in the mathematics classroom as well as the conceptions and beliefs of the members of that community.

The teaching-experiment methodology applied to a whole class (as opposed to a focus on an individual learner) is particularly needed to contribute to a reform agenda that calls for major changes in the way mathematics is thought of and taught in this country. As mathematics, mathematics learning, and mathematics teaching are reconceptualized, we must question to what extent past research on mathematics classrooms is relevant. Most important, we need to reexamine whether effective teaching, as defined by extant research, has much to say about classrooms that reflect these new conceptions. The classroom teaching experiment, as described in this volume, is research that is based, not on status-quo instruction, but rather on what the authors call following Richards's (1991) "inquiry mathematics," an approach to mathematics instruction that reflects the direction of current reforms. If we are able to understand the implications of current reform efforts, more studies need to be conducted that are based on mathematics instruction as it is envisioned rather than as it is currently practiced.

Steffe (1983) asserts that the development of models is the most important aspect of the teaching experiment. At this point in efforts to reform mathematics education, we must create models that are useful in explaining the psychological and sociological processes that can work interactively to result in students developing powerful mathematical ideas. It would be premature to translate envisioned reform into actual reform and omit the key step of developing such models. This point is emphasized by Hunting (1983):

> A priori specification and consequent experimental manipulation of variables or "environments" assumed to explain mathematical behavior can presume too much in the absence of well developed models of human intellectual functioning. (pp. 45–46)

The classroom teaching experiment as employed by Cobb, Wood, and Yackel has several essential elements for contributing to informed change in mathematics education: real classroom context, instruction guided by envisioned reform and recent research, psychological and sociological analysis, an approach that allows the researcher to learn about which interventions might be appropriate and which effects to observe, and the resulting development of theoretical models for learning in inquiry-mathematics classrooms. The teaching experiment allows researchers to pursue the important question "How does it work?" rather than the less elucidating "Does it work?"

The research also contributes to available methodology by identifying important units of analysis for studying mathematics classrooms. In Chapter 3, the authors point out that "a classroom mathematics tradition is in many ways analogous to a scientific research tradition ... both are created by a community and both influence individuals' construction of scientific or mathematical knowledge by constraining what can count as a problem, a solution, an explanation, and a justification...." The qualitative study of the nature of problem solutions, explanations, and justifications in a mathematics classroom is likely to provide a useful way of characterizing mathematics learning and teaching interactions. Additionally, the researchers have demonstrated the need to focus on taken-as-shared meanings that are constituted in order to understand the mathematical activity in the classroom. Their study of classroom discourse has led to characterization of two types of discourse: discourse about mathematics and discourse about doing and talking about mathematics. These categories seem to be particularly helpful to our understanding of how new classroom mathematics traditions might be constituted.

CONTRIBUTION MADE BY THE RESEARCH RESULTS

The research of Cobb, Wood, and Yackel offers an existence proof with respect to proposed reforms in elementary school mathematics instruction. The research reported in this monograph demonstrates that second graders can devise solution strategies, justify their solutions, and critique the solutions of others. It suggests the effectiveness of a combination of collaborative group problem solving and whole-class discussion. Although much has been written about the need for change in mathematics instruction and proposed directions for that change, few other studies have carefully documented the effect of these reforms on the mathematics tradition that is currently well established in elementary school classrooms.

In the inquiry classroom, the roles of teacher and student are different from those they would play in the traditional classroom. The teacher no longer acts as the source of information but rather as the chief architect of the learning opportunities in the classroom, a facilitator of whole-class discussion, and an ongoing assessor of student learning. Teacher-designated learning activities become raw material for the classroom activities that are mutually constituted by teacher and students. This characterization emphasizes both the interactive determination of classroom activities and the asymmetric roles of teacher and students in this process. Although the classroom activities are shaped by both the teacher and the students, their experience, knowledge, means of interaction, and power relationship are not equivalent. (For a more detailed discussion of the teacher's role in inquiry classrooms, see Simon, 1986).

One of the striking and very important aspects of the work described in this volume is that the researchers have taken on a real classroom with all the inherent issues of making change in a public school context. They accepted the school district's curriculum demands and are working with a teacher who has the typical demand of teaching the full elementary school curriculum. She, in turn, is working with second-grade students who have already learned to expect traditional instruction in the classroom. Thus, the researchers were able to focus not only on what inquiry mathematics looks like in the classroom but also on how it becomes established. In this respect, the researchers have demonstrated the essential role played by classroom organization and discourse. Classroom mathematics experiences establish expectations as to what constitutes mathematical activity in the classroom and contributes to the students' conceptions of mathematical activity in general. Thus, being asked to explain an idea or to justify a solution not only reveals the students' underlying reasoning, it also contributes to establishing the idea that explanation and justification are part of doing mathematics in the classroom community.

ISSUES ARISING FROM INTERPRETATIONS OF THE DATA

I wish to offer alternative interpretations for two aspects of the reported work. The first issue involves Wood's identification of a tension that exists for the teacher in inquiry-mathematics instruction. The following quote from the introduction to Chapter 6 focuses my discussion: "On the one hand she [the teacher] wanted to encourage children's individual mathematical constructions, but on the other hand she wanted them to be acculturated into the mathematical practices of society." From my own studies of inquiry-mathematics instruction, I would suggest that the major tension experienced by teachers is somewhat different from, but perhaps overlaps with, the one just described.

Teachers in inquiry classrooms necessarily constrain the mathematical nature and direction of the students' work through the tasks that they set and the questions that they ask. Without such constraints, students are not likely to construct many of the powerful mathematical ideas that have been developed historically. The key tension, then, for teachers is between responding to the mathematical constructions, questions, and interests of the student on the one hand and following their own agenda on the other. This agenda is informed by the mathematics of the larger culture as well as many other factors (e.g., beliefs about learning in general, the particular mathematics in question, and the expectations of the school district). The difficulty is not that the mathematics created by the students is for the most part in conflict with the mathematics of the larger culture; it is that teachers must strive to make decisions that balance appropriately the spontaneous contributions of the students and their sense of structuring the situation for optimal learning. Thus, the tension is less one of conflicting mathematics and more one of providing structure and direction versus following student initiatives.

The second issue involves the development of inquiry mathematics in the classroom. Cobb, Wood, and Yackel's research has significantly contributed to our understanding of how inquiry mathematics is mutually constituted through dialogue involving teacher and students. This represents the sociological lens. It is essential to the analysis of any mathematics lesson that the knowledge and perspectives of all the individuals involved are characterized as effectively as possible. However, just as the construction of mathematical ideas has an individual, psychological component, so does the construction of what it means to do and to learn mathematics in this classroom community. Teachers' developing concepts of mathematics, learning, and teaching affect the instructional decisions that they make and the nature of the interaction that they have with their students (see Simon, 1991; Simon & Schifter, 1991).

The teacher in this second-grade teaching experiment was just beginning to learn about inquiry mathematics. Her schemes for making sense of what she was doing and the skills she brought to teaching were derived from both her recent experience with inquiry mathematics and her prior experience with traditional school mathematics. In order to more fully understand the interactions among students and teacher, the researchers must be able to develop a useful characterization of the teacher's relevant knowledge and beliefs at that point in time. Instead of seeing her instructional practice as an example of the teacher's role in inquiry mathematics, it might be more appropriate to see her actions as reflecting her development from one mathematics tradition to another. As such, some of her actions support the growth of an inquiry-mathematics tradition in the classroom, whereas others maintain aspects of traditional school mathematics. The interaction between Justin and the teacher toward the end of Chapter 6 provides a specific case to examine.

Recall that Justin and Sally believed that 32 take away 23 is 11. In this section, Wood writes,

> Thus she [the teacher] continues the dialogue in a directive manner to reduce the ambiguity in their thinking.
>
> *Teacher:* All right, put up two fingers (Justin puts up two fingers). Now take away three of them. Can't do that, can you? So the way you were doing that is not working (pause). What is 23 and 10?
>
> *Justin:* 33.
>
> *Teacher:* 33, and we only have 32 seats and 23 people sitting in them (pause). Do you still agree with 11?
>
> *Justin:* You get 9 out of it.

The analysis of this section ends with the statement, "She waits and then asks, 'Do you still agree with 11?' " This creates for Justin an opportunity to rethink his solution, to reconsider the validity of his procedure, and to make the final comment."

From my perspective, it seems reasonable that this is an example of a teacher in an inquiry classroom reverting to a more traditional approach in response to a difficult situation, that of two students whose arithmetical misconception is persisting in the context of a problem-solving discussion. The fact that the teacher attempts to lead Justin through her thinking may in fact be evidence that the teacher has not completely given up the notion that she can replace the student's thinking with her own. Even though this may be a practice she avoids in most cases, her final question, "Do you still agree with 11?" and some of her earlier comments such as, "You can't do that, can you?" and "So the way you were doing that is not working" may have pressured Justin into recanting his solution of 11 and accepting the class's answer of 9. The interaction seems to end after Justin has said, "You get 9 out of it." He has offered no evidence that he understands why 9 is correct nor why his prior thinking was incorrect.

Returning to the more general point, we have found (Simon, 1989) that teachers who are changing their instruction represent interesting and unique mixes of inquiry mathematics and traditional mathematics conceptions. At any particular point in time, in what ways do the teachers' conceptions constrain classroom mathematical practices? To begin to address this question, researchers must focus more directly on the teachers' developing knowledge and the sense that they make of classroom interactions. This point is discussed further in the next section under recommendations for future work.

EXTENDING AND BUILDING ON THIS RESEARCH

I see three areas for extending this research, which has a great deal of potential for mathematics education.

1. *Study of the development of particular mathematical ideas in the inquiry classroom.* There is a need in inquiry-mathematics classrooms to focus on the development of particular mathematical concepts and ideas by individual students and by students collectively in an inquiry-mathematics classroom. Such work would not only contribute to our understanding of learners' development of these ideas, but would also give us insight into how the inquiry tradition contributes to the development of specific mathematical ideas. The research reported here focuses more on the nature of the mathematics activity (e.g., problem solving, justification). Although this informs us as to how certain ideas become taken-as-shared, this focus does not give us a clear sense of the development, over time, of powerful mathematical ideas. The tracing of the development of important mathematical

understandings is likely to provide a valuable window on the impact of an inquiry-mathematics tradition.

2. *Extension of the research to upper grades.* The picture provided by this work on inquiry mathematics in the second-grade classroom whets our collective appetite for a view of inquiry mathematics in upper elementary and secondary mathematics classrooms. Our work at Mount Holyoke College (Simon, 1989; Simon & Schifter, 1991) suggests that expanding such efforts into the upper grades increases in many ways the complexity of the problem. The following is a brief discussion of some aspects of this complexity.

- The research base on mathematics learning is less developed than that available for first- and second-grade mathematics. The research reported in this monograph relied on cognitive models developed by Carpenter, Hiebert, and Moser (1983), Kamii (1985), Steffe, et al. (1988), and Steffe, et al. (1983);
- Most second-grade mathematics is learned by all students, at least by the time they finish elementary school. When the curriculum focuses on mathematical ideas that many people never understand, (e.g., fractions, ratio, function), instructional challenges grow considerably.
- Older students who have not understood the mathematics they studied have significant gaps in their mathematical knowledge that may go back several years. The high school mathematics teacher who is attempting to teach algebra, geometry, or calculus is often confronted with students who do not understand fractions, ratio, or the distributive property. (In an Algebra II class that I was observing, a test was returned and one of the students turned to the student next to her and said, "If you know what the score is, how do you figure out the percent?") The dilemma for the teacher in this situation is whether to teach the mathematics that is described in the curriculum or to go back and work on the shaky mathematical foundation that is preventing meaningful learning of the prescribed curriculum.
- Students in the upper elementary years and beyond have been socialized for many years in a school mathematics tradition. Initially, many students feel threatened in an inquiry classroom by demands that they develop their own mathematical ideas, justify those ideas, and critique the ideas of others. Teachers frequently hear, "Just tell me how to do it." The renegotiation of social norms therefore is a much greater problem with students who have participated in school mathematics for a longer time. Change in instruction in the lower grades could eventually eliminate this problem. Elementary teachers who are moving toward an inquiry tradition often report that they have changed their way of teaching in other areas besides mathematics (Simon & Schifter, 1991). This consistency throughout the academic day probably supports the estab-

lishment of new classroom norms. The upper elementary and secondary school students, who see a different teacher for each curriculum area, are less likely to encounter instruction in other areas that is consistent with what occurs in their mathematics classes.

- The mathematics taught by upper elementary and secondary school teachers is more complex than the mathematics taught by primary school teachers. As a result, it is more common that upper grade teachers do not thoroughly understand the mathematics that they teach. I would suggest that the teaching of upper grade mathematics requires not only deeper conceptual understanding but also a broader conception of the nature of mathematics. Much of the arithmetic of whole numbers can be situated within the demands of the everyday lives of most people. As the curriculum begins to include topics such as ratio, negative numbers, algebra, and complex numbers, a more comprehensive view of the nature of mathematics is required.

3. *Careful study of the interaction between teacher development and the culture of the classroom.* This point follows directly from the discussion of the preceding section. Research that looks carefully at the teacher's evolvement toward an inquiry-mathematics tradition is essential for two reasons. First, it is important to understand the relationship between what teachers believe or know and the role they play in constituting the mathematics classroom culture. Understanding the teacher's perspective as well as the students' is an essential part of understanding classroom interaction. Although cognitive models of young children's mathematical development informed Cobb, Wood, and Yackel's interpretation of children's mathematical behavior, corresponding models of teachers' development (in the direction of inquiry mathematics) have not yet been developed. Our Construction of Elementary Mathematics Project has begun to study such development in prospective elementary school teachers.

A second reason for studying teacher development in this context is that given the profound nature of the changes that are envisioned by the reform in mathematics instruction, it is important to understand how teachers learn to teach in these new ways. For example, is there a pattern to how their conceptions evolve? The understanding of these learning processes is essential to promoting effective change. Our work has indicated that a wide range exists in teachers' conceptions following extensive teacher development opportunities. Certainly, inquiry mathematics will not be successfully implemented by all teachers who participate in such programs. Careful study of teachers as they transform their practice and of their developing conceptions may help us analyze the elements of successful change. Study in this area requires characterizing the nature of mathematical activity in classrooms and how it influences students' learning. The work of Cobb, Wood, and Yackel represents a significant contribution to a methodology that meets this demand.

Chapter 9

CONTEXT FOR CHANGE: THEMES RELATED TO MATHEMATICS EDUCATION REFORM

Martin Simon

Just as the social context of the classroom contributes to and constrains the mathematical activity of individuals (discussed in Chapter 8), the school and district community likewise influences the character of the classrooms within it. In order to understand the process of mathematics education reform in schools, it is necessary to view the reform as embedded in the larger community. Deborah Dillon has provided us with an ethnographic look at the sociopolitical context for the changes in the second-grade mathematics classroom that are described in this monograph. The story that she developed from her diverse sources of data has many themes that can inform future innovations of this type.

I chose four particular types of themes to discuss in this chapter. The first is the political realities that are encountered in reform of the type described. Understanding the larger context involves understanding the goals, motivations, and typical ways of functioning for the different constituents who become involved. My second and third topics of discussion are the issues that arise when one engages in changing paradigms and the inherent vulnerability of innovative programs. These topics suggest that we must also understand the mathematics reform in the larger context of instructional change in general. Finally, I consider the important issue of teacher empowerment that is raised by reform, whether or not that reform is promoted by the teachers involved.

POLITICAL REALITIES

Several issues in this story reveal political patterns that affect the implementation of innovative programs. One is the key role of school and district administrators in supporting reform efforts. Administrators frequently experience a tension between their educational objectives and political appeasement. Although some administrators aspire to be educational leaders who move their districts forward toward improved instruction, there is very little reward and often substantial risk for administrators who associate themselves with educational innovations. One example is the assistant superintendent for instruction in the school district involved in this study. Initially, he supported the problem-solving approach to mathematics. He withdrew this support when the first shot was fired from one member of the school board and in fact joined in the efforts to limit the use of the second-grade project. This example highlights the important distinction between administrative consent and administrative commitment to defend innovative programs. It is the latter that is important to the survival of innovative programs. Unfortunately, such commitment is rarely garnered.

Another theme that emerged is the importance of proactive rather than reactive communication with parents, administrators, and board members regarding the innovative program. Hindsight, of course, is far clearer than foresight. Although there were some efforts initially to present parent workshops and classes were open for observation to both school and district administrators, no effort to communicate with the community and with the administration was undertaken until the program was clearly threatened. The lesson is that the interests, traditions, and views of the community and the administration must be considered and channels of communication actively pursued in order to inform community members and administrators and to preempt potential conflicts.[1]

One of the key issues that emerges in Dillon's description is determining who has the power and who has the expertise to make decisions regarding instruction. What are the respective roles of the school board, administrators, teachers, and parents in these decisions? It seems reasonable to assume that the school district described in this volume is not unusual in its lack of clarity about such questions. In this particular scenario, although the teachers are probably the most qualified to make decisions with respect to which program of mathematics instruction to use (certainly they are the most familiar with the project's approach), they were the least influential group in setting the direction of the second-grade mathematics program once the school board, parents, and administrators became involved. This involvement centered on perceptions of the effectiveness of the program. Innovators in mathematics education must wrestle with and negotiate how the innovative program will be evaluated, who will do the evaluating, and who will translate the evaluation into decision making. Additionally, this discussion brings us to issues of teacher empowerment, which are discussed later in this chapter.

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1. Besides the potential political benefits of having parents, administrators, and school members well informed and possibly inclined toward the innovative program, there are direct benefits to students. Although it is not talked about in this volume, it is my experience that uninformed parents confronted with their children's unfamiliar mathematical activities often make comments that undermine students' confidence in their mathematics instruction, for example: "I don't know what they are showing you in class. Let me just show you an easy way to do that." In the case of a comment of this type, which was motivated only by parents' desire to help their children, the parents inadvertently have two undermining effects. First, they have provided children with a rote procedure at a time when the teacher has been working hard to have students think about mathematics and develop their own strategies. Second, they have set up a conflict in children between their loyalty to their parents and to their teachers. If students feel that parents do not respect their learning of mathematics in school, often their natural loyalties will cause them to resist the program and to devalue it. It is therefore, essential that innovative effors of this type work to encourage supportive feelings on the part of parents and, if possible, to give them appropriate helpful ways to respond at home to their child's work.

CHANGING PARADIGMS

Changing paradigms for mathematics instruction is fraught with difficulties that go beyond the classroom. How is it that administrators, parents, and community members, who have an interest in what goes on in the classroom, come to support and even believe in a new approach to mathematics instruction? On the one hand, the traditional paradigm that underlies the thinking of this group identifies mathematics as a series of skills and facts that if learned, lead to efficient and accurate calculations. Students' learning, from this view involves listening and reading carefully followed by large amounts of repetitive practice. Doing mathematics is essentially a solitary activity, and teachers, through textbooks, determine the validity of particular mathematics.

The new paradigm, on the other hand, holds that mathematics is a way of thinking and that it involves problem solving, communication, and conceptual understanding. Mathematical knowledge is constructed by individuals and groups through exploration, hypothesis, verification, explanation, justification, and negotiation of meaning. In this view, mathematics is done by individuals, small groups, and large groups. Mathematical validity is determined by the mathematical community.

So where are the common values that could be the basis of dialogue between those working from the old paradigm and those who are promoting the new one? I suggest the following points of intersection. Each group involved wants students to be successful in mathematics, to be confident, and to enjoy their mathematical experience. All are concerned with mathematics learning, the ability to apply learned mathematics, and the accessibility of advanced mathematics courses for fields of study that require advanced mathematics.

It is imperative that educators who are attempting to institute a new paradigm speak the language of those who still rely on the traditional paradigm. Communication with parents and administrators can only be useful when the information given to them is related to their concerns. It is the extent to which they feel their concerns are being met that determines the initial support of these constituencies. George Brown (1974), speaking to educators involved in a different educational reform movement, said, "We [had] better go out and learn their language because they are sure not going to learn ours." Speaking the language of these constituencies is not just a matter of verbal expression, it is also a matter of structuring aspects of the program so that the groups involved are less likely to feel threatened or uncomfortable and so they find aspects of the program that are familiar. An example of this may be to assign homework on a regular basis that is reassuring to the parents yet not in direct conflict with the goals of the program.

Dillon's story points to the necessity for effective and regular communication. A well-informed community is an asset to an innovative program, preventing misunderstandings and the passing along of misinformation. Hearing about a program from the educators involved is more likely to foster support than hearing about it from a disgruntled parent or board member. However, the issue is larger than sending letters home or conducting parent meetings to explain the program. Although these are both useful strategies, we must be aware that educators who have come to understand mathematics instruction in a way that is consistent with the inquiry approach have done so as a result of a complex set of experiences with mathematics, with children's learning, and with teaching. Such understandings are not "passed along" through a description of the program (any more than mathematical understandings can be directly transmitted to the students). Although educators cannot provide the community with sufficient experience to fully understand the program, they can attempt to involve the community in some useful ways. Whenever possible, communications with administrators, other teachers, parents, and community members should be experiential in nature. This can involve participating in hands-on mathematics workshops, viewing classroom videos, making visits to project classrooms, and viewing videos of children solving problems in interview settings. Such experiences can help in establishing understanding of the educational problems and of the program that is being implemented as a possible solution.

VULNERABILITY OF INNOVATIVE PROGRAMS

Significant change in mathematics instruction requires an understanding by all involved that innovative efforts tend to go through an awkward stage and that initial bugs will have to be worked out. New programs thus depend on a greater tolerance for difficulties than existing programs do. Unfortunately, the opposite is usually the case. Innovative programs are extremely vulnerable. People who are already skeptical of the new efforts may be especially sensitive to difficulties experienced in the program, which to them may be a sign that the efforts are misdirected. Dillon's story documents the reaction of one family whose child scored very poorly on the state-mandated achievement test and their attribution of his low score to the change in mathematics instruction. From her description, there seems to be evidence that the child's difficulties were not a result of the change in the second-grade approach to mathematics instruction but were consistent with his poor achievement in all subjects.

The tendency of administrators and community members to attribute any difficulties to the change in instruction can be understood in the following way. Difficulties that arise in educational settings tend to be complex and difficult to analyze. The presence of an innovative program can be a convenient scapegoat for many difficulties that are otherwise difficult to explain. It is unlikely that those who do not actively support the innovation will analyze the situation carefully to establish an appropriate baseline for the traditional program against which the new program can be compared. As a result, the innovative program becomes a lightning rod for discontent.

TEACHER EMPOWERMENT

I wish to end my commentary by considering two questions: (1) What is the appropriate role of teachers in choosing instructional approaches to subject matter? (2) Given that teachers had in the past been told what mathematics program they would use, why were the administrative and board decisions with respect to the project particularly upsetting to the second-grade teachers?

Ideally, teachers are the members of the community who are best prepared to make instructional decisions. They have participated in professional education, they have direct experience in working with students, and it is their business to evaluate on an ongoing basis the impact of the instruction they provide. However, actually the majority of elementary school teachers of mathematics are not well prepared to be the instructional leaders in their communities. Providing such leadership necessitates being able to make instructional decisions that can be defended in terms of the mathematical content being taught and in terms of conceptions of students' learning about mathematics. Unfortunately, most teachers have not had the opportunity or the encouragement to develop and articulate theories of mathematics learning and often do not thoroughly understand the mathematics they teach. The latter is less of a problem with teachers in the first and second grade, who usually understand the mathematics they teach. However, it was only through the teachers' experience in the second-grade project that they developed a useful understanding of their students' thinking processes and abilities to learn mathematics. Having paid close attention to children's learning for one year and having experimented with strategies that promote learning in important ways, these teachers were now prepared to make instructional choices and to justify those choices in ways that they were not able to do before the project. Having reached this level of competence and feeling the confidence that comes with such professional growth, it was no longer acceptable to these teachers to have the district dictate the approach to mathematics instruction that they would use. Whereas prior to the project the teachers used the latest textbook or tried to follow the district's view of effective mathematics instruction, they now had a sound and personally developed basis for making instructional decisions on their own. Thus, their power as professionals had increased. However, their political power, the power to influence educational decisions, had not increased commensurately. It is essential that there be a balance between empowerment that comes from professional competence and the power that comes from political clout.[2] It is only through promoting the two together that we can improve educational leadership in school districts.

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[2] Increasing teachers' power to shape instructional policy without providing them with the opportunity to increase their competence to do so is likewise inadequate. Thus, hiring teacher volunteers during the summer to write a new curriculum often falls short of effecting significant change.

CONCLUSION

The research of Cobb, Wood, Yackel, and Dillon has provided perhaps the most detailed look at mathematics education reform in elementary school currently available. It has emphasized the complementary and interactive nature of the three levels of activity: activity by the individual class member, activity within the classroom community, and activity within the larger sociopolitical contexts. Thus, it is important that future studies combine psychological, sociological, and anthropological analyses, as has been done successfully in this study. The researchers have provided theory that can be built on and methodologies that will evolve through adaptation in subsequent research. It is only through detailed study of this type that we will come to understand the nature of mathematics reform, the challenges that must be met, and what it will take to meet these challenges.

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